MA 3205 – Set Theory – Homework due Week 5

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Homework. The homework follows the lecture notes. You have to hand in one starred homework in Weeks 3–6, in Weeks 7–9 and in Weeks 10–13. Further homework can be checked on request. Homework to be marked should be handed in after the lecture on Tuesday of the week when the homework is due.

Exercise 5.6*. Determine the functions f_n given by the following recursive equations:

1. $f_1(0) = 0$, $f_1(S(n)) = f_1(n) + 2^n$,
2. $f_2(0) = 1$, $f_2(1) = 0$, $f_2(n+2) = f_2(n) \cdot \frac{4n+2}{n+1}$,
3. $f_3(n) = 1$ for $n = 0, 1, \ldots, 9$, $f_3(10n + m) = f_3(n) + 1$ for $n = 1, 2, \ldots$ and $m = 0, 1, \ldots, 9$,
4. $f_4(0) = 0$, $f_4(1) = 0$, $f_4(2) = 0$, $f_4(3) = 1$, $f_4(S(n)) = f_4(n) + \frac{1}{2}(n^2 - n)$ for $n > 2$,
5. $f_5(n) = 1$, $f_5(S(n)) = 256 \cdot f_5(n)$.

Give informal explanations what these functions compute, for example, consider f_6 given by $f_6(0) = 0$, $f_6(1) = 0$ and $f_6(S(n)) = f_6(n) + 2n$ for $n \geq 1$. Then $f_6(n) = n(n-1)$. One explanation would be to assume that there is a soccer league with n teams. Then there are $f_6(n)$ games per season, each pair $\{A, B\}$ of two different teams plays once at A's place and once at B's place. Another example would be coming from set theory; let the function $f_7(n)$ say how many subsets of $n = \{0, l, \ldots, n-1\}$ have an even number of elements. Then f_7 is given by $f_7(0) = 1$, $f_7(1) = 1$ and $f_7(n+1) = 2 \cdot f_7(n)$ for all $n > 0$.

Exercise 5.9. Let $H : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ be a function and $h_m : \mathbb{N} \to \mathbb{N}$ be given by $h_m(n) = H(m, n)$ for all n. Show that there is a function f dominating every h_m; that is, show that there is a function f such that for every m there is an n with $f(k) > h_m(k)$ for all $k > n$.

Exercise 6.7. Prove by giving a one-to-one function that the set {Auckland, Christchurch, Dunedin, Wellington} of New Zealand's largest towns has a cardinality which is less than the set {Adelaide, Brisbane, Canberra, Melbourne, Perth, Sydney} of Australian towns. Furthermore, prove that it is not less or equal than the cardinality of the set {Singapore}.

Exercise 6.11. Show that if $|X| = |X \times \mathbb{N}|$ then $|\{0, 1\}^X| = |\mathbb{N}^X|$.