MA 3205 – Set Theory – Homework due Week 7

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Homework. The homework follows the lecture notes. You have to hand in one starred homework in Weeks 3–6, in Weeks 7–9 and in Weeks 10–13. Further homework can be checked on request. Homework to be marked should be handed in after the lecture on Tuesday of the week when the homework is due.

Note that the first midterm test is on Tuesday 15 September 2009 in the first half of the lecture. Please revise the first 8 chapters of the lecture notes for this midterm test.

Exercise 9.7. Let $A = \mathbb{N} - \{0, 1\} = \{2, 3, 4, \ldots\}$ and let $<_{div}$ be given by $x <_{div} y \Leftrightarrow \exists z \in A\,(x \cdot z = y)$. That is, $x <_{div} y$ iff x is a proper divisor of y, so $2 <_{div} 8$ but $2 \not<_{div} 2$ and $2 \not<_{div} 5$. Prove that $(A, <_{div})$ is a partially ordered set.

Exercise 9.10*. Prove that the following relations are partial orderings on $\mathbb{N}^{\mathbb{N}}$:

- $f \sqsubset_1 g \Leftrightarrow \exists n\, \forall m > n\, (f(m) < g(m))$;
- $f \sqsubset_2 g \Leftrightarrow \forall n\, (f(n) \leq g(n)) \wedge \exists m\, (f(m) < g(m))$;
- $f \sqsubset_3 g \Leftrightarrow \forall n\, (f(n) \leq g(n)) \wedge \exists n\, (f(n) < g(n)) \wedge \exists n\, \forall m > n\, (f(m) = g(m))$;
- $f \sqsubset_4 g \Leftrightarrow f(0) < g(0)$.

Determine for every ordering a pair of incomparable elements f, g such that neither $f \sqsubset_m g$ nor $g \sqsubset_m f$ nor $f = g$. For which of these orderings is it possible to choose the f of this pair (f, g) of examples such that $f(n) = 0$ for all n?

Exercise 9.14. Let $A = \mathbb{N} - \{0, 1\}$ and $<_{div}$ given by $x <_{div} y \Leftrightarrow \exists z \in A\,(x \cdot z = y)$ as in Exercise 9.7. Define a relation E on $A \times A$ by putting (x, y) into E iff there is a prime number z with $x \cdot z = y$. So $(2, 4) \in E$, $(2, 6) \in E$, $(2, 10) \in E$ but $(2, 7) \notin E$, $(2, 8) \notin E$ and $(2, 20) \notin E$. Show that $(A, <_E)$ and $(A, <_{div})$ are identical partially ordered sets.