Exercise 17.12. Consider the following partial ordering given on the set $\mathbb{N}^\mathbb{N}$ of all functions from $\mathbb{N}$ to $\mathbb{N}$:

$$f \sqsubseteq g \iff \exists n \forall m > n (f(m) < g(m)).$$

This partial ordering only shares some but not all of the properties of the ordering $<_{\text{lin}}$ considered in the lecture. In order to see this, show the following two properties:

- For countably many functions $f_0, f_1, \ldots$ there is a function $g$ such that $\forall n \in \mathbb{N} (f_n \sqsubseteq g)$;
- There are uncountably many $f$ below the exponential function $n \mapsto 2^n$. Namely for every $A \subseteq \mathbb{N}$ the function $c_A : n \mapsto \sum_{m \in n} 2^{n-m-1} \cdot A(m)$ is below the exponential function.

Note that $c_A \sqsubseteq c_B \iff A <_{\text{lex}} B$. Thus there is an uncountable linearly ordered set of functions below the exponential function.

Exercise 17.13*. Use the Axiom of Choice to prove the following: If $|A| = \aleph_1$ and every $B \in A$ satisfies $|B| \leq \aleph_1$ then $|\bigcup A| \leq \aleph_1$.

Exercise 18.9. Verify that Hausdorff’s axioms are true for the set $\mathbb{R}$. That is, verify that $(\mathbb{R}, \{A \subseteq \mathbb{R} \mid A \text{ is open}\})$ is a Hausdorff space.

Exercise 18.10. Let $\alpha$ be any ordinal. Define a topology on $\alpha$ by saying that $\beta$ is open iff $\beta$ is an ordinal and $\beta \subseteq \alpha$. Verify that the first three axioms of Hausdorff are satisfied, but not the last fourth one.

Exercise 18.11. Find a topology on the set of ordinals up to a given ordinal $\alpha$ which satisfies the Axioms of Hausdorff and in which an ordinal $\beta \in \alpha$ is isolated iff it is either a successor ordinal or $0$. 

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