5.1* Models. (a) Let $X$ contain the following formulas:

- $\exists x [P x]$;
- $\forall x, y, v, w [(P x \land P y \land P v \land P w \land (x \neq v \lor y \neq w)) \rightarrow f(x, y) \neq f(v, w)]$;
- $\forall u, v, w [P v \land P w \rightarrow \neg P(f(v, w))]$;
- $\forall u \exists v, w [\neg P u \rightarrow (P v \land P w \land f(v, w) = u)]$.

Assume that there is a finite set $A$ with a function $f : A \rightarrow A$ and a predicate $P$ on $A$ is given which is a model of $X$. Let $n$ be the number of elements $a \in A$ satisfying $Pa$ and $m$ be the number of elements $a \in A$ satisfying $\neg Pa$. How do $m$ and $n$ relate to each other?

(b) Make a set $Y$ of formulas using a function $f$ and predicate $P$ such that the $m, n$ from above satisfy $m = 1 + 2 + 3 + \ldots + n$.

5.2 Graphs. A graph $G$ is a base set $V$ with a relation $E$ such that $E(x, y)$ stands for $x$ and $y$ being connected in the graph. A graph $(V, E)$ is called a random graph if $V$ is infinite and for every two finite disjoint sets $C, D$ of vertices there is a vertex $z$ such that $E(x, z)$ for all $x \in C$ and $\neg E(y, z)$ for all $y \in D$. Make a set $X$ of formulas such that a graph $(V, E)$ satisfies $X$ iff $(V, E)$ is a random graph.

5.3 Matrix Rings. Construct a set $X$ of formulas which enforces that a structure $(R_1, R_2, +, \cdot, 0, 1, det, e_0, e_1, e_2, e_3)$ has the following properties: $(R_1, +, \cdot, 0, 1)$ is a commutative ring with 1 and is a subring of a noncommutative ring $(R_2, +, \cdot, 0, 1)$ with $R_1 \subseteq R_2$ such that $(R_2, +, \cdot, 0, 1)$ is isomorphic to the ring of $2 \times 2$-matrices over $R_1$ and $det$ assigns to every member of $R_2$ the value which the determinant over the corresponding matrix would have. $0$ and $1$ are the neutral elements for ring addition and ring multiplication in $R_2$. The constants $e_0, e_1, e_2, e_3$ can be used freely to define the structure.

5.4 Dense Linear Orders. Assume that $(A, <, 0, 1)$ is a linearly ordered set satisfying the additional formulas $0 < 1$ and $\forall x, y \exists z [0 \leq x < z < y \leq 1 \lor 0 \leq y < z < x \leq 1 \lor 0 \leq x = z = y \leq 1]$. Recall that $x \leq y$ abbreviates $x < y \lor x = y$; furthermore, linear orders satisfy the axioms $\forall x, y, z [x < y \land y < z \rightarrow x < z]$ and $\forall x, y [x < y \lor y < x \land x = y]$ and $\forall x [\neg x < x]$. Show that all countable models of this ordering are isomorphic (provided that preassignments to free variables are not taken into account).