MA 5219 - Logic and Foundations of Mathematics 1
Homework due in Week 6, Tuesday.

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Hand in each starred homework; 1 mark per homework (if it is correct), up to 10 marks in total for homework.

6.1 Substituion. (a) Make the following substitutions of formulas ϕ standing for $\forall x(x \circ y = y)$ and ψ standing for $\exists x(x \circ y = z)$: $(\phi)\frac{1}{x}$, $(\phi)\frac{y \circ 1 \circ 0}{y}$ $(\psi)\frac{y}{z}\frac{z}{y}$, $((\phi \rightarrow \psi)\frac{y \circ y}{z})\frac{2}{y}$.
(b) A question left over from the lecture is why it would be a problem to substitute a free variable by a term containing a bound variable. For this, assume that the the structure given are the natural number and discuss what would happen if one would do the substitutions like $(\exists x\,[y = x])\frac{x+1}{y}$. Give also an example where a false formula becomes true by such a substitution.

6.2* Models and Compactness. Assume that the underlying logical language is infinite and contains the constants $c_0, c_1, \ldots$, the predicates $P_0, P_1, \ldots$ and the variable x. Furthermore, let X be a set of formulas containing the formulas $P_n(c_n)$ and $\neg P_n(c_m)$ for all n and all $m \neq n$. Find a set Y of open formulas of the form $P_n(t)$ and $\neg P_n(t)$ where t is a term such that for every $F \subseteq Y$ the following is true:

- If F is finite then there is a model $\mathcal{M}$ with base set A such that $\mathcal{M} \models X \cup F$ and for every $a \in A$ there is a constant c_n with $c_n = a$;
- If $F = Y$ then for every model $\mathcal{M}$ with base set A there is an $a \in A$ with $c_n \neq a$ for all constants c_n.

6.3 Rings. Let $(A, +, \cdot, 0, 1, a, b)$ be a ring satisfying the following set X of formulas:
$a \cdot a = a \wedge b \cdot b = b \wedge 0 \neq 1$;
$\forall x, y, z\,[x + y = y + x \wedge (x + y) + z = x + (y + z)]$;
$\forall x, y, z\,[x \cdot y = y \cdot x \wedge (x \cdot y) \cdot z = x \cdot (y \cdot z)]$;
$\forall x, y, z\,[x \cdot (y + z) = (x \cdot y) + (x \cdot z)]$;
$\forall x\,[x + 0 = x \wedge x \cdot 1 = x]$;
$\forall x \exists y\,[x + y = 0]$;
$\forall x \exists y, z\,[x = a \cdot y + b \cdot z]$;
$\forall x \exists y\,[x \cdot a = 0 \vee (x \cdot a) \cdot (y \cdot a) = a]$;
$\forall x \exists y\,[x \cdot b = 0 \vee (x \cdot b) \cdot (y \cdot b) = b]$;
Find the least number n such that $n \geq 2$ and n cannot be the cardinality of A.