MA 5219 - Logic and Foundations of Mathematics 1
Homework due in Week 10, Tuesday.

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Hand in each starred homework; 1 mark per homework (if it is correct), up to 10 marks in total for homework.

10.1* Truth in Arithmetics.
Mojzesz Presburger introduced the axioms of arithmetic with addition which are the following:

1. $x + 1 \neq 0$;
2. $x + 1 = y + 1 \rightarrow x = y$;
3. $x + 0 = x$;
4. $x + (y + 1) = (x + y) + 1$;
5. If ϕ is a first-order formula using addition and constants $0, 1, \ldots$ and some variables including z and if x, y do not occur in ϕ then the following formula is an axiom: $\phi\frac{0}{z} \wedge \forall x\, [\phi\frac{x}{z} \rightarrow \phi\frac{x+1}{z}] \rightarrow \phi\frac{y}{z}$.

Note that in 5., the formula ϕ itself can have quantified parts and bounded variables different from x, y, z. These axioms are today referred to as Presburger arithmetic. Mojzesz Presburger proved that Presburger arithmetic is consistent, complete and decidable. Say that a function f is definable in Presburger arithmetic iff there is a formula ϕ with two free variables in Presburger arithmetic satisfying $\forall x \forall y\, [y = f(x) \leftrightarrow \phi(x, y)]$. Which of the following functions is definable in Presburger arithmetic?

1. $f_1(x) = 3 \cdot x + 5$;
2. $f_2(x) = x^2$;
3. $f_3(x) = round(x/2)$ where $round(y)$ is the largest natural number below y: $round(0) = 0$, $round(2.5) = 2$ and $round(3) = 3$;
4. $f_4(x) = 1$ if x is odd and $f_4(x) = 0$ if x is even.

If the function or relation is definable then give the corresponding formula else say in a few words why it is undefinable.

10.2 Register Programs.
Write a register program computing the recursive function f given by the following formulas (with $x > 0$): $f(0) = 0$; $f(1) = 0$; $f(2x) = f(x) + 1$; $f(2x + 1) = f(x) + 1$.