MA 5219 - Logic and Foundations of Mathematics 1
Homework due in Week 10, Tuesday.

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Hand in each starred homework; 1 mark per homework (if it is correct), up to 10 marks in total for homework.

13.1 Rice’s Theorem. Let $\varphi_0, \varphi_1, \ldots$ be any G"odel numbering of the partial-recursive functions with one input. Say two programs $d,e$ are equivalent iff for all inputs $x$, either $\varphi_d(x)$ and $\varphi_e(x)$ are both undefined or they are both defined and equal. Now $E$ is said to be an index-set iff for all equivalent indices $d,e$, either both $d,e \notin E$ or both $d,e \in E$. Rice’s Theorem says that there are no recursive index-set besides $\emptyset$ and $\mathbb{N}$. For which of the following sets, one can use Rice’s Theorem to show that they are not recursive (none of these sets is actually recursive):

1. $\{e : \varphi_e(0) \text{ is defined}\}$;
2. $\{e : \varphi_e(e) \text{ is defined}\}$;
3. $\{e : \varphi_e(x) \text{ is defined for infinitely many } x\}$;
4. $\{e : \text{ at least one of } \varphi_e(256), \varphi_e(257) \text{ and } \varphi_e(258) \text{ is defined}\}$;
5. $\{e : \varphi_e(0) \text{ and } \varphi_e(\varphi_e(0)) \text{ are both defined}\}$.

13.2 Hierarchies of Sets. A subset $A \subseteq \mathbb{N}$ is called a $\Sigma_n$-set iff there exists an arithmetic $\Sigma_n$-formula $\phi$ with $A = \{n \in \mathbb{N} : (\mathbb{N},+,\cdot,0,1) \models \phi(n)\}$. Similarly one defines $\Pi_n$-sets. An alternative approach is the following: An r.e. set is a $\Sigma_1$-set and a co-r.e. set is a $\Pi_1$-set. Then, a $\Sigma_{n+1}$-set is any set which can be enumerated by a function $f$ which is recursive relative to $A$ where $A$ is a $\Sigma_n$-set. Here “recursive relative to $A$” means that $f$ is computed by a register program which uses $A$ as a subprogram to determine whether $A(x)$ is 1 ($x \in A$) or 0 ($x \notin A$). Use the fact that the two definitions of $\Sigma_n$-sets coincide to prove that there is no formula $\phi$ with one parameter such that $\phi(n)$ is true iff the $n$-th sentence (according to some G"odel numbering of all sentences) is true in $(\mathbb{N},+,\cdot,0,1)$. This gives an alternative proof of Tarski’s theorem.

13.3 Nonexistence of Decidable Extensions of $Q$. Assume by way of contradiction that there is a theory $T$ extending $Q$ for which the set $\{gn(\phi) : \phi \text{ is a sentence and } \vdash_T \phi\}$ is decidable. Then show that there is a partial-recursive $\{0,1\}$-valued function $\psi$ such that no total recursive function $f$ extends $\psi$. Produce a formula $\phi$ such that $\psi(n) = 0$ implies $\phi(n)$ and $\psi(n) = 1$ implies $\neg \phi(n)$. Use this to derive a contradiction with the assumptions on $T$. Here $gn(\phi)$ is the G"odel number of the formula $\phi$. 
