Hand in each homework which you want to be checked; 1 mark per each correct starred homework; up to 10 marks in total for homework - there will be more than 10 starred homeworks, so you have several chances to try.

4.1 Proof systems. Consider the following rules.

\[
\begin{array}{c}
\emptyset \vdash A \\
X \vdash A \\
X \vdash A \rightarrow B \\
X, X \cup Y \vdash A \\
X, A \vdash B \\
X, A \vdash B \rightarrow C \\
X \vdash A \rightarrow B \\
X \vdash A \rightarrow C \\
X \vdash A \rightarrow B \\
\end{array}
\]

Derive the following rules from the above rules.

\[
\begin{array}{c}
X \vdash A \rightarrow B, \neg A \vdash B \\
X \vdash A \rightarrow B, X, A \vdash B, \neg A \vdash B \\
\end{array}
\]

4.2\*: Operators. Is there a structure \((A, \circ)\) satisfying the law of commutativity and the below law of inversion though \(\circ\) does not need to have a neutral element, that is, \((A, \circ)\) satisfies only the first of the following two laws:

\[
\begin{align*}
\forall a, b \in A & \exists c \in A \quad [a \circ b = b \circ a \land a \circ c = b]; \\
\exists c \in A & \forall a \in A \quad [a \circ c = a \circ e = a]?
\end{align*}
\]

Note that \(\circ\) is not required to be associative. Prove your answer.

4.3\*: Ordered Semigroups. Assume that \((A, \circ, \leq)\) is an ordered semigroup such that \(\circ\) is associative, \(\leq\) is transitive and the following two rules hold:

\[
\begin{align*}
\forall a, b \in A & [a \leq b \land b \leq a \Leftrightarrow a = b]; \\
\forall a, b, c \in A & [a \leq b \Rightarrow a \circ c \leq b \circ c \land c \circ a \leq c \circ b].
\end{align*}
\]

Does every model of such a semigroup have an element \(b\) with \(\forall a \in A [a \circ b = b \circ a]?)?