8.1 Logical Implication. Here let $e$ is a constant and $v, w, x, y, z$ are variables. Let $X$ contain

$$
\forall x, y, z \left[ x \circ (y \circ z) = (x \circ y) \circ z \right], \ \forall x \left[ x \circ e = x \right], \ \forall x \left[ x \circ x = e \right]
$$

$\alpha$ be $v \circ w = w \circ v$,

$\beta$ be $v \circ (v \circ v) = v$ and

$\gamma$ be $v \circ (v \circ v) = e$.

Do $X \models \alpha, X \models \beta$ and $X \models \gamma$ hold? Justify for each of the formulas $\alpha, \beta, \gamma$ your answers.

8.2 Henkin Sets. Let the logical language $\mathcal{L}$ contain first-order formulas over variables $x_0, x_1, \ldots$ and constants $c_0, c_1, \ldots$ and one predicate symbol $P$. Check whether the following sets $X$, $Y$ and $Z$ are Henkin sets. Explain why you think that the corresponding sets are Henkin sets or not.

$X$ contains for all distinct $i, j$ the formula $c_i \neq c_j$ as well as for every $k$ the formulas $x_k = c_k$ and $P(c_k)$.

$Y$ contains the same formulas as $X$ plus the formula $\forall x_0 \left[ P(x_0) \right]$.

$Z$ contains the same formulas as $X$ plus the formula $\exists x_0 \left[ \neg P(x_0) \right]$.

8.3* Countable models. Assume that $X$ is a Henkin set and $\mathcal{A}$ is a model of $X$. Show that $\mathcal{A}$ has a substructure $\mathcal{B}$ which is an at most countable model of $X$.

8.4* Proof System. Use the enlarged rule system on next page obtained by adding Modus Ponens and two similar further rules for $\rightarrow$ to those of page 92 in order to prove the following tautology (where $e$ is a constant and $\circ$ an operation symbol which could be rewritten as a function with two inputs):

$$
\forall x \left[ x \circ (x \circ x) = e \right] \rightarrow \forall x, y, z \left[ x \circ (y \circ z) = (x \circ y) \circ z \right] \rightarrow \forall x \left[ x \circ e = x \right] \rightarrow \\
\forall x \left[ e \circ x = x \right] \rightarrow \forall y, z \left[ y \circ (z \circ (y \circ (z \circ y))) = z \circ z \right].
$$
Derivation Rules. Here are the rules from page 92 of Rautenberg’s book plus the three rules for →.

\( (IR) \quad \frac{}{X \vdash \alpha} \quad \text{where } \alpha \in X \text{ or } \alpha \text{ is of the form } t = t \)

\( (MR) \quad \frac{}{X \cup Y \vdash \alpha} \)

\( (MP) \quad \frac{X \vdash \alpha, \alpha \rightarrow \beta}{X \vdash \beta} \)

\( (\rightarrow 1) \quad \frac{X \vdash \alpha \rightarrow \beta}{X, \alpha \vdash \beta} \)

\( (\rightarrow 2) \quad \frac{X \vdash \alpha}{X, \alpha \vdash \beta} \)

\( (\wedge 1) \quad \frac{}{X \vdash \alpha, \beta} \)

\( (\wedge 2) \quad \frac{X \vdash \alpha \wedge \beta}{X \vdash \alpha, \beta} \)

\( (\neg 1) \quad \frac{X \vdash \beta, \neg \beta}{X \vdash \alpha} \)

\( (\neg 2) \quad \frac{X, \beta \vdash \alpha \mid X, \neg \beta \vdash \alpha}{X \vdash \alpha} \)

\( (\forall 1) \quad \frac{X \vdash \forall x(\alpha)}{X \vdash \alpha^{(\frac{t}{x})}} \quad \text{where } \alpha, \frac{t}{x} \text{ is collision-free} \)

\( (\forall 2) \quad \frac{X \vdash \alpha^{(\frac{y}{x})}}{X \vdash \forall x(\alpha)} \quad \text{where } y \not\in free(X) \cup var(\alpha) \)

\( (=) \quad \frac{X \vdash s = t, \alpha^{(\frac{s}{x})}}{X \vdash \alpha^{(\frac{t}{x})}} \quad \text{where } \alpha \text{ is prime} \)