MA 5219 - Logic and Foundations of Mathematics 1
Course-Webpage `http://www.comp.nus.edu.sg/~fstephan/mathlogic.html`
Homework due in Week 10, Tuesday 22 October 2013.

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Hand in each homework which you want to be checked; 1 mark per each correct starred homework; up to 10 marks in total for homework - there will be more than 10 starred homeworks, so you have several chances to try.

10.1* Birkhoff rules. Adjust the axioms of the Birkhoff rules to incorporate binary associative operations like $\circ$ as follows (where the fourth rule is given for a function f of arity 3 but applies for functions of all arities):

$$\frac{\emptyset}{t=t} \qquad \frac{s=t}{t=s}$$

$$\frac{s=t, t=r}{s=r} \qquad \frac{s_1=t_1, s_2=t_2, s_3=t_3}{f(s_1,s_2,s_3)=f(t_1,t_2,t_3)}$$

$$\frac{\emptyset}{(r\circ s)\circ t = r\circ s\circ t} \qquad \frac{\emptyset}{r\circ (s\circ t) = r\circ s\circ t}$$

$$\frac{s=t}{s\circ r = t\circ r} \qquad \frac{s=t}{r\circ s = r\circ t}$$

$$\frac{s=t}{q\circ s\circ r = q\circ t\circ r} \qquad \frac{s=t}{s^\sigma = t^\sigma} \text{ for all global substitutions } \sigma$$

In the following let x,y,z be variables and d,e be constants.
(a) Assume now that the additional axioms $x\circ x\circ x\circ x\circ x\circ x = e$, $e\circ x = x$, $x\circ e = x$ and $d\circ d\circ d\circ d\circ d = e$ are given. Prove that $d = e$.
(b) Assume now that the additional axioms $x\circ x\circ x\circ x\circ x\circ x = e$, $e\circ x = x$, $x\circ e = x$ and $d\circ d\circ d = e$ are given. Prove that one can neither derive $d = e$ nor derive $x\circ y = y\circ x$ by producing a model in which these two formulas are false.
(c) Assume now that the additional axioms $x\circ d = d$, $x\circ e = x$, $e\circ x = x$, $f(x\circ y) = f(y)\circ f(x)$, $f(f(x)) = x$ and $x\circ x\circ f(x) = x$ are given, where f is a unary function symbol. Derive $f(x)\circ x\circ x = x$. Furthermore, provide a model which shows that $f(x) = x$ cannot be derived.

10.2* Second Order Logic. Second order language permits to quantify over sets. Using that the there are uncountably many subsets of the natural numbers, give a set X of formulas such that every second order model of X is uncountable. X should of course be satisfiable, that is, have at least one model. It is permitted to introduce constant symbols, predicate symbols and function symbols; furthermore, the symbol $\in$ in the formula $a \in B$ is true iff a is an element of B; lower case variables range over elements and upper case variables range over sets.