MA 4207 - Mathematical Logic
Homework due in Week 3.

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Homework 3.1
Cantor’s function \( x, y \mapsto (x + y) \cdot (x + y + 1)/2 + y \) is a bijection from \( \mathbb{N} \times \mathbb{N} \) onto \( \mathbb{N} \).
Construct a bijection from \( \mathbb{Z} \times \mathbb{Z} \) onto \( \mathbb{Z} \).

Homework 3.2
Prove that there is no set \( X \) such that its powerset \( \{ Y : Y \subseteq X \} \) has 5 elements.

Homework 3.3
Show that a power set has always more elements than the given set, that is, fill out the missing details at the following proof-sketch. Recall that \( |A| \leq |B| \) iff there is a one-one function from \( A \) to \( B \) and show that \( |2^A| \nleq |A| \).

Proof-Sketch: The \( \emptyset \) has 0 and \( 2^\emptyset \) has one element, namely \( \emptyset \), hence one cannot have a one-one mapping from \( 2^\emptyset \) to \( \emptyset \). Now assume that \( A \) is not empty and \( f : A \rightarrow 2^A \) is a function. Show that there is a set \( B \subseteq A \) which is not in the range of \( f \). Then consider any function \( g : 2^A \rightarrow A \) and prove that this function cannot be one-one, as otherwise a surjective \( f \) from \( A \) to \( 2^A \) would exist. Hence \( |2^A| \nleq |A| \).

Homework 3.4
Use Homework 3.3 to prove that there is no set \( X \) such that its powerset has as many elements as \( \mathbb{N} \). The fact that every set \( X \) is either finite or satisfies \( |\mathbb{N}| \leq |X| \) can be used in the proof.

Homework 3.5
Let \( f(n) \) be the maximum number of negation symbols in a well-formed formula which does not contain any subformula of the form \( \neg(\neg\alpha) \) and which contains at most \( n \) atoms. Here \( \neg(A_1 \lor (\neg(A_2 \lor (\neg(A_1)))) \) has 3 atoms and \( n \) is 3, as repeated atoms are counted again. Determine the value \( f(n) \) in dependence of \( n \).

Homework 3.6
Prove by induction that a well-formed formula of length \( n \) contains less than \( n/3 \) connectives and at most \( (n + 3)/4 \) atoms.

Homework 3.7
Use the truth-table method to prove that the following formulas are equivalent:

- \( ((\neg A_1) \lor (\neg A_2)) \);
- \( (\neg(A_1 \land A_2)) \);
- \( ((A_1 \lor A_2) \leftrightarrow (A_1 \oplus A_2)) \).
**Homework 3.8**
List out the truth-table for the formula \((A_1 \oplus A_2) \land (\neg A_3)\).

**Homework 3.9**
Consider the following formulas:

\[
\begin{align*}
\phi_1 & = (((A_1 \lor A_2) \lor A_3) \land ((A_4 \lor A_5) \lor A_6)); \\
\phi_2 & = (((A_1 \lor A_2) \land (A_3 \lor A_4)) \land (A_5 \lor A_6)); \\
\phi_3 & = (((((A_1 \oplus A_2) \oplus A_3) \oplus A_4) \oplus A_5) \oplus A_6).
\end{align*}
\]

There are \(2^6 = 64\) ways to assign the truth-values to the sentence symbols (or atoms) \(A_1, \ldots, A_6\). Determine for each of the formulas \(\phi_1, \phi_2, \phi_3\), how many of these assignments make the formula true and how many of these assignments make the formula false.

**Homework 3.10**
For the formulas from Homework 3.9, is the statement

\[
\{\phi_1, \phi_2, \phi_3\} \models (((A_1 \land A_2) \land A_3) \land A_4) \land A_5) \land A_6
\]

true or false? Prove your answer.

**Homework 3.11**
For the formulas from Homework 3.9, is the statement

\[
\{\phi_1, \phi_2, \phi_3\} \models ((((A_1 \lor A_2) \lor A_3) \lor A_4) \lor A_5) \lor A_6)
\]

true or false? Prove your answer.

**Homework 3.12**
Using the connectives \(\lor, \land, \rightarrow, \leftrightarrow, \oplus, \neg\), construct a formula using atoms \(A_1, A_2, A_3, A_4\) which says that at least two and at most three of these atoms are true.

**Homework 3.13**
Using the connectives \(\lor, \land, \rightarrow\), construct a formula using atoms \(A_1, A_2, A_3, A_4, A_5, A_6\) which says that at all six atoms are either false or all six atoms are true.