MA 4207 - Mathematical Logic

Homework due in Week 4.

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Homework 4.1
Let $\text{atom}(\phi)$ be the set of atoms used in $\phi$, so $\text{atom}(((A_1 \lor A_2) \land A_2)) = \{A_1, A_2\}$ and $\text{atom}((0 \lor 1)) = \emptyset$. Let WFF be the set of well-formed formulas. Let $C_1 = \{\phi \in \text{WFF} : \forall v [v(A) = 1 \text{ for some } A \in \text{atom}(\phi) \text{ then } v(\phi) = 1]\}$.

For which of the connectives $\neg, \land, \lor, \rightarrow, \leftrightarrow, \oplus$ is $C_1$ closed under the connective? Here one says that $C$ is closed under the connective $\oplus$ if all formulas $\phi, \psi \in C$ satisfy that $(\phi \oplus \psi) \in C$. Similarly for other connectives.

Homework 4.2
Let $\text{atom}(\phi)$ and WFF be defined as in Homework 4.1. Let $C_2 = \{\phi \in \text{WFF} : \forall v [v(A) = 0 \text{ for at most one } A \in \text{atom}(\phi) \text{ then } v(\phi) = 1]\}$.

For which of the connectives $\neg, \land, \lor, \rightarrow, \leftrightarrow, \oplus$ is $C_2$ closed under the connective?

Homework 4.3
Let $C_3 = \{\phi \in \text{WFF} : \text{every } A \in \text{atom}(\phi) \text{ occurs in } \phi \text{ exactly once and } 0, 1 \text{ do not occur in } \phi\}$. Prove by induction that $C_3$ does not contain any tautology and also not contain any antitautology. Here a tautology is a formula which is always true (independent of the choice of the truth-values of the atoms) and an antitautology is a formula which is always false.

Homework 4.4
Define on WFF by recursion the functions $\text{atom}(\phi) \mapsto \text{atom}(\phi)$ and $\text{maxatom}(\phi) \mapsto \max\{k : A_k \in \text{atom}(\phi)\}$, where the atoms $A_1, A_2, \ldots$ can be used and $\max\emptyset = 0$. So $\text{maxatom}((A_1 \lor (A_4 \land A_5))) = 5$ and $\text{maxatom}((0 \lor 1)) = 0$.

Homework 4.5
Define on WFF by recursion the function $\text{numcon}(\phi)$ as the number of connectives $\land, \lor, \rightarrow, \leftrightarrow, \oplus$ occurring in $\phi$. Furthermore define the function $\text{numneg}(\phi)$ as the number of negations occurring in $\phi$. Determine the best-possible constants $c, m, n$ such that

$$|\phi| \leq c \cdot \text{numcon}(\phi) + n \cdot \text{numneg}(\phi) + m$$

for all WFF $\phi$.

Homework 4.6
Let $C_6 = \{\phi \in \text{WFF} : \text{every } A \in \text{atom}(\phi) \text{ occurs in } \phi \text{ exactly once}\}$; note that formulas in $C_6$ might have occurrences of the constants 0 and 1. Define by recursion a function $F$ from $C_6$ into the rational numbers between 0 and 1 which returns for
each formula $\phi \in C_6$ the truth-probability $n/2^m$ where $n$ is the number of rows in the truth-table of $\phi$ evaluated to 1 and $m$ is the number of atoms used in the formula so that $2^m$ is the overall number of rows in the truth-table of $\phi$. For example, $F(1) = 1$, $F((A_1 \oplus (A_2 \lor A_3))) = 1/2$ and $F(((A_2 \lor A_3) \land (A_3 \lor 0))) = 3/8$.

**Homework 4.7**
Let $C_7 = \{ \phi \in WFF : \phi$ can use the constants 0, 1 and the only connectives in $\phi$ are $\land$ and $\lor$}. Prove by induction that a formula $\phi \in C_7$ is a tautology iff $v(\phi) = 1$ for the truth-assignment $v$ with $v(A_k) = 0$ for all $k$.

**Homework 4.8**
Let $C_8 = \{ \phi \in WFF : \phi$ can use the constants 0, 1 and the only connectives in $\phi$ are $\land$ and $\lor$}. Prove by induction that a formula $\phi \in C_8$ is an antitautology iff $v(\phi) = 0$ for the truth-assignment $v$ with $v(A_k) = 1$ for all $k$.

**Homework 4.9**
Let $U$ be a finite set of atoms and $C_9 = \{ \phi \in WFF : \text{atom}(\phi) \subseteq U \}$. Prove that there is a finite set $F$ of formulas such that for every formula $\phi \in C_9$ there is a $\psi \in F$ with $(\psi \leftrightarrow \phi)$ being a tautology.

**Homework 4.10**
The following formulas have brackets omitted according to the rule that the binding strengths of the connectives is ordered as $\neg$, $\land$, $\lor$, $\oplus$, $\rightarrow$, $\leftrightarrow$. Insert back the needed brackets for getting a member of WFF.

1. $A_1 \land \neg A_2 \lor A_3 \rightarrow A_4 \land \neg A_5$;
2. $A_1 \lor \neg A_2 \land \neg A_3 \leftrightarrow A_4 \rightarrow A_5$;
3. $\neg \neg A_1 \lor \neg A_2$.

**Homework 4.11**
Is there a formula using the connectives $\oplus$ and $\neg$ but no other connectives where the value of the formula depends on the placement of brackets?

**Homework 4.12**
Let $v(A_1) = 1$, $v(A_2) = 1$, $v(A_3) = 0$. The below formulas are given in Polish notation. Write them as WFF and evaluate them according to $v$:

1. $\neg \leftrightarrow A_1 A_2 A_3$;
2. $\land \lor \neg \lor A_1 A_2 A_3 A_4$;
3. $\oplus \land A_1 A_2 \land A_2 A_3$.

**Homework 4.13**
Write the following formula in Polish notation: $\neg((A_1 \lor \neg A_2) \land (A_2 \lor \neg A_3) \land (A_3 \lor \neg A_4))$. 
