MA 4207 - Mathematical Logic
Course-Webpage `http://www.comp.nus.edu.sg/~fstephan/mathlogicug.html`
Homework due on Friday of Week 8 (and Tuesday subsequent week).

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Homework 8.1
Let $(A, +, *)$ and $(B, +, *)$ be the remainder rings modulo a and b, respectively, $a, b \in \{2, 3, 4, 5, 6\}$. For which a, b is there a homomorphism f from $(A, +, *)$ to $(B, +, *)$ such that any two terms t_1, t_2 satisfy $(A, +, *) \models t_1 = t_2$ iff $(B, +, *) \models f(t_1) = f(t_2)$.

Homework 8.2
Choose values for a, b from Homework 8.1 and a formula ϕ and a function f such that f is a homomorphism and the formula ϕ is true in $(A, +, *)$ but not in $(B, +, *)$.

Homework 8.3
Consider the model $(\{0, 1, \ldots, 9\}, +, \cdot)$ with addition and multiplication modulo 10, so $5 + 7 = 2$ and $5 \cdot 7 = 5$. Which are the sets defined by the following formulas:

1. $x \in A \Leftrightarrow \exists y\,[x = y \cdot y]$;
2. $x \in B \Leftrightarrow \forall y\,[x \cdot y = 0 \vee x \cdot y = 3 \vee x \cdot y = 5]$;
3. $x \in C \Leftrightarrow \forall y\,[x \neq y \cdot y \cdot y \cdot y]$.

Homework 8.4
Is the set $\{2, 4, 6, 8\}$ definable in the model of arithmetic modulo 10? Here the formula can use the operations $+, \cdot$ and the constants $0, 1$ and $=, \neq$ and quantifiers.

Homework 8.5
Is every function in the model $\{0, 1, \ldots, 9\}$ with addition and multiplication and all constants explictly definable by a term? If so, give a proof; if not, explain why.

Homework 8.6
Let $\mathbb{Z} * \{i\} + \mathbb{Z}$ be the set of all complex integer numbers. Show that this set together with $+$ and $*$ is a ring. Prove that the basis element i is not definable by using an isomorphism which maps i to some other element.

Homework 8.7
Recall that a structure $(A, \circ, e)$ is a group iff it satisfies $\forall x, y, z\,[x \circ (y \circ z) = (x \circ y) \circ z]$, $\forall x\,[x \circ e = x \wedge e \circ x = x]$, $\forall x \exists y\,[x \circ y = e \wedge y \circ x = e]$.
Write down formally the axioms for an Abelian group, a ring with 1 and a commutative ring with 1, respectively.

Homework 8.8
Assume that $(A, +, *, 0, 1)$ is a finite ring with $0 \neq 1$. Consider the formulas

$$x = 0 \quad \Leftrightarrow \quad \forall y\,[x + y = y] \text{ and}$$
$$x = 1 \quad \Leftrightarrow \quad \forall y\,[x * y = y \wedge y * x = y].$$

Are then all members of A definable with formulas like this? If yes then prove how this is done else provide a finite ring where some elements are not definable.

Homework 8.9
Let $(\mathbb{R}, +, *, <, 0, 1)$ be the ordered field of the real numbers with the constants 0 and 1. Prove that all rational numbers and all real roots of polynomials are definable. Provide then examples of formulas $\phi_1, \phi_2, \phi_3, \phi_4$ such that x_k is the unique element satisfying ϕ_k where the formulas ϕ_k say the following:

1. $x_1 = 2/3$;
2. x_2 is the positive square-root of 3;
3. x_3 is the largest number satisfying $x_3^{10} - 4x_3^5 + 2 = 0$;
4. x_4 is the smallest number satisfying $3x_4^6 - 6x_4^4 + 3x_4^2 = 0$.

Homework 8.10
Consider a structure $(A, f, 0, 1, =)$ with $0, 1 \in A$ being constants and f a function from A to A. Make three different formulas in the language of this structure which express the following respective conditions:

1. f has the range $\{0, 1\}$;
2. f is the inverse of itself;
3. every value in the range of f is the image of exactly two values.

Homework 8.11
Let $(A, P^A), (B, P^B)$ be two structures with $A = \{0\}$ and $B = \{1, 2\}$. choose the predicate P^B such that there is no strong homomorphism from B to A (independently of what P^A is) while there is for each possible choice of P^A a strong homomorphism from (A, P^A) to (B, P^B).

Homework 8.12
Let $(\mathbb{Z}, Succ, Even)$ be a structure with $Even(x)$ being true iff x is even and $Succ$ being the successor function. Let f be a function from the structure to itself. Prove that if f is a homomorphism then f is a strong homomorphism.

Homework 8.13
Consider the structure $(\mathbb{Z}, Neigh, Even)$ where $Even(x)$ is true iff x is even and $Neigh(x, y)$ is true iff $x = y + 1$ or $x = y - 1$. Construct a function g from $\mathbb{Z}$ to itself which is a homomorphism but not a strong homomorphism.