Homework due on Friday of Week 8 (and Tuesday subsequent week).

Homework 8.1
Let \((A,+,\cdot)\) and \((B,+,\cdot)\) be the remainder rings modulo \(a\) and \(b\), respectively, \(a,b \in \{2,3,4,5,6\}\). For which \(a,b\) is there a homomorphism \(f\) from \((A,+,\cdot)\) to \((B,+,\cdot)\) such that any two terms \(t_1,t_2\) satisfy \((A,+,\cdot) \models t_1 = t_2 \iff (B,+,\cdot) \models f(t_1) = f(t_2)\).

Homework 8.2
Choose values for \(a,b\) from Homework 8.1 and a formula \(\phi\) and a function \(f\) such that \(f\) is a homomorphism and the formula \(\phi\) is true in \((A,+,\cdot)\) but not in \((B,+,\cdot)\).

Homework 8.3
Consider the model \(\langle \{0,1,\ldots,9\},+,\cdot \rangle\) with addition and multiplication modulo 10, so 5 + 7 = 2 and 5 · 7 = 5. Which are the sets defined by the following formulas:

1. \(x \in A \iff \exists y \ [x = y \cdot y]\);
2. \(x \in B \iff \forall y \ [x \cdot y = 0 \lor x \cdot y = 3 \lor x \cdot y = 5]\);
3. \(x \in C \iff \forall y \ [x \neq y \cdot y \cdot y \cdot y]\).

Homework 8.4
Is the set \(\{2,4,6,8\}\) definable in the model of arithmetic modulo 10? Here the formula can use the operations +,· and the constants 0, 1 and =, ≠ and quantifiers.

Homework 8.5
Is every function in the model \(\{0,1,\ldots,9\}\) with addition and multiplication and all constants explicitly definable by a term? If so, give a proof; if not, explain why.

Homework 8.6
Let \(\mathbb{Z} \star \{i\} + \mathbb{Z}\) be the set of all complex integer numbers. Show that this set together with + and \(\cdot\) is a ring. Prove that the basis element \(i\) is not definable by using an isomorphism which maps \(i\) to some other element.

Homework 8.7
Recall that a structure \((A,\circ,e)\) is a group iff it satisfies \(\forall x,y,z \ [x \circ (y \circ z) = (x \circ y) \circ z]\), \(\forall x \ [x \circ e = x \land e \circ x = x]\), \(\forall x \exists y \ [x \circ y = e \land y \circ x = e]\).
Write down formally the axioms for an Abelian group, a ring with 1 and a commutative ring with 1, respectively.
Homework 8.8
Assume that \((A, +, *, 0, 1)\) is a finite ring with \(0 \neq 1\). Consider the formulas
\[
\begin{align*}
x &= 0 & \iff & \forall y [x + y = y] \quad \text{and} \\
x &= 1 & \iff & \forall y [x * y = y \land y * x = y].
\end{align*}
\]
Are then all members of \(A\) definable with formulas like this? If yes then prove how this is done else provide a finite ring where some elements are not definable.

Homework 8.9
Let \((\mathbb{R}, +, *, <, 0, 1)\) be the ordered field of the real numbers with the constants 0 and 1. Prove that all rational numbers and all real roots of polynomials are definable. Provide then examples of formulas \(\phi_1, \phi_2, \phi_3, \phi_4\) such that \(x_k\) is the unique element satisfying \(\phi_k\) where the formulas \(\phi_k\) say the following:
\[
\begin{align*}
1. \quad x_1 &= 2/3; \\
2. \quad x_2 &= \text{the positive square-root of 3}; \\
3. \quad x_3 &= \text{the largest number satisfying } x_3^{10} - 4x_3^5 + 2 = 0; \\
4. \quad x_4 &= \text{the smallest number satisfying } 3x_4^6 - 6x_4^4 + 3x_4^2 = 0.
\end{align*}
\]

Homework 8.10
Consider a structure \((A, f, 0, 1, =)\) with 0, 1 \(\in A\) being constants and \(f\) a function from \(A\) to \(A\). Make three different formulas in the language of this structure which express the following respective conditions:
\[
\begin{align*}
1. \quad &f\text{ has the range } \{0, 1\}; \\
2. \quad &f\text{ is the inverse of itself}; \\
3. \quad &\text{every value in the range of } f \text{ is the image of exactly two values}.
\end{align*}
\]

Homework 8.11
Let \((A, P^A), (B, P^B)\) be two structures with \(A = \{0\}\) and \(B = \{1, 2\}\). choose the predicate \(P^B\) such that there is no strong homomorphism from \(B\) to \(A\) (independently of what \(P^A\) is) while there is for each possible choice of \(P^A\) a strong homomorphism from \((A, P^A)\) to \((B, P^B)\).

Homework 8.12
Let \((\mathbb{Z}, \text{Succ}, \text{Even})\) be a structure with \(\text{Even}(x)\) being true iff \(x\) is even and \(\text{Succ}\) being the successor function. Let \(f\) be a function from the structure to itself. Prove that if \(f\) is a homomorphism then \(f\) is a strong homomorphism.

Homework 8.13
Consider the structure \((\mathbb{Z}, \text{Neigh}, \text{Even})\) where \(\text{Even}(x)\) is true iff \(x\) is even and \(\text{Neigh}(x, y)\) is true iff \(x = y + 1\) or \(x = y - 1\). Construct a function \(g\) from \(\mathbb{Z}\) to itself which is a homomorphism but not a strong homomorphism.