Homework 11.1
Assume that $\alpha, \beta, \gamma$ are well-formed formulas. Give a formal proof of the statement

\[
\{\beta, \gamma\} \models \alpha \rightarrow \beta
\]

which only uses the formulas from $\Lambda$ and the Modus Ponens.

Homework 11.2
If $\{\alpha, \beta\}$ tautologically implies $\gamma$, is then the below derivation correct? Explain your answer.

1. $\{\alpha, \beta\} \vdash \alpha \rightarrow \beta \rightarrow \gamma$ (Axiom Group 1)
2. $\{\alpha, \beta\} \vdash \alpha$ (Copy)
3. $\{\alpha, \beta\} \vdash \beta \rightarrow \gamma$ (Modus Ponens)
4. $\{\alpha, \beta\} \vdash \beta$ (Copy)
5. $\{\alpha, \beta\} \vdash \gamma$ (Modus Ponens)

For the following exercises, $P, Q$ are predicates and $a, b, c$ are constants.

Homework 11.3
Make a formal proof for

\[
\{\forall x \[P(x) \rightarrow Q(c)\], \forall x \[\neg P(x) \rightarrow Q(c)\]\} \vdash Q(c)
\]

Homework 11.4
Make a formal proof for $\{\forall x \[P(x)\], \exists y \[\neg P(y)\]\} \vdash Q(z)$.

Homework 11.5
Make a formal proof for $\emptyset \vdash \forall x \forall y \[P(x) \rightarrow Q(y)\] \rightarrow P(a) \rightarrow Q(b)$.

Homework 11.6
Is the statement $\emptyset \vdash P(x) \rightarrow \forall y \[P(y)\]$ correct? Explain your answer.

Homework 11.7
Is the statement $\emptyset \vdash P(x) \rightarrow \forall y \[P(x)\]$ correct? Explain your answer.
Homework 11.8
Is the statement \( \emptyset \vdash P(x) \rightarrow \exists y [P(y)] \) correct? Explain your answer.

Homework 11.9
Let \((G, \circ, f, e)\) be a structure and \(\Gamma\) contain the following axioms:

- \(\forall x, y, z [(x \circ y) \circ z = x \circ (y \circ z)]\);
- \(\forall x, y [x \circ y = y \circ x]\);
- \(\forall x [x \circ e = x]\);
- \(\forall x [x \circ f(x) = e]\);
- \(\forall x, y, z [x \circ y = x \circ z \rightarrow y = z]\);

So \((G, \circ)\) is an Abelian group with neutral element \(e\) and inversion \(f\). Prove informally the following results:

- \(\forall v, w [f(v) = f(w) \rightarrow v = w]\);
- \(\forall v, w [v \circ w = e \rightarrow f(v) = w]\);
- \(\forall v, w [f(v \circ w) = f(w) \circ f(v)]\).

Homework 11.10
Consider all structures \((A, \circ)\) where \(A\) has two elements and satisfies the axioms

\[\forall x [x \circ x = x] \text{ and } \forall x \forall y [x \circ y = y \circ x].\]

Show that all these structures are isomorphic.

Homework 11.11
Assume that \((\mathbb{N}, +, <, 0, 1, P)\) is a structure where \(\mathbb{N}\) is the set of natural numbers and +, <, 0, 1 have the usual meaning on \(\mathbb{N}\). Let the powers of 2 be the set \(\{1, 2, 4, 8, 16, \ldots\}\) and make a formula \(\alpha\) such that \((\mathbb{N}, +, <, 0, 1, P) \models \alpha\) iff \(\forall x [P x \leftrightarrow x \text{ is a power of } 2]\).

Note that such a formula only implicitly defines the powers of 2 and not explicitly; therefore this formula \(\alpha\) does not say that the powers are definable from addition and order in \(\mathbb{N}\).

Homework 11.12
Make a formula \(\alpha\) which says that \(f : A \rightarrow A\) is a one-to-one function but not an onto function. Provide a model \((A, f, =)\) which satisfies \(\alpha\). Can \(A\) be finite?