**MA 4207 - Mathematical Logic**
Course-Webpage `http://www.comp.nus.edu.sg/~fstephan/mathlogicug.html`
Homework

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**Homework 2.1**
Cantor's function $x, y \mapsto (x+y) \cdot (x+y+1)/2 + y$ is a bijection from $\mathbb{N} \times \mathbb{N}$ onto $\mathbb{N}$. Construct a bijection from $\mathbb{Z} \times \mathbb{Z}$ onto $\mathbb{Z}$.

**Homework 2.2**
Prove that there is no set $X$ such that its powerset $\{Y : Y \subseteq X\}$ has five elements.

**Homework 2.3**
Show that a power set has always more elements than the given set, that is, fill out the missing details at the following proof-sketch. Recall that $\mathrm{Card}(A) \leq \mathrm{Card}(B)$ iff there is a one-one function from $A$ to $B$ and show that $\mathrm{Card}(\mathbb{P}(A)) \not\leq \mathrm{Card}(A)$.

Proof-Sketch: The $\emptyset$ has 0 and $\mathbb{P}(\emptyset)$ has one element, namely $\emptyset$, hence one cannot have a one-one mapping from $\mathbb{P}(\emptyset)$ to $\emptyset$. Now assume that $A$ is not empty and $f : A \to \mathbb{P}(A)$ is a function. Show that there is a set $B \subseteq A$ which is not in the range of $f$. Then consider any function $g : \mathbb{P}(A) \to A$ and prove that this function cannot be one-one, as otherwise a surjective $f$ from $A$ to $\mathbb{P}(A)$ would exist. Hence $\mathrm{Card}(\mathbb{P}(A)) \not\leq \mathrm{Card}(A)$.

**Homework 2.4**
Use Homework 2.3 to prove that there is no set $X$ such that $\mathrm{Card}(\mathbb{P}(X)) = \aleph_0$. The fact that every set $X$ is either finite or satisfies $\aleph_0 \leq \mathrm{Card}(X)$ can be used in the proof.

**Homework 2.5**
Determine the cardinality of the set $\{X \subseteq Y : \mathrm{Card}(X) = 3\}$ for each of the following sets $Y$: (a) $Y = \{0, 1\}$; (b) $Y = \{0, 1, 2, 3, 4\}$; (c) $Y = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$; (d) $Y = \mathbb{N}$.

**Homework 2.6**
Consider the set $\mathbb{D} = \{q : q$ is a rational number with $0 \leq q < 1$ such that its denominator is a power of $2\}$. Construct a bijection from $\mathbb{N}$ to $\mathbb{D}$ explicitly.

**Homework 2.7**
A set $X$ is well-ordered iff every non-empty subset $Y \subseteq X$ contains a minimal element, that is, an element $z \in Y$ with $z < u$ for all $u \in Y - \{z\}$. A set $X$ is strongly well-ordered iff every non-empty subset $Y \subseteq X$ contains a minimal element and a maximal element. Determine for which cardinals $\kappa$ there is a strongly well-ordered set $X$ with $\mathrm{Card}(X) = \kappa$.

**Homework 2.8**
Consider the following sets $X$ and $Y$ and show that they satisfy $\mathrm{Card}(X) = \mathrm{Card}(Y)$ by constructing explicitly a bijection $g$ from $X$ to $Y$. Here

$$\begin{aligned} X &= \{f : \mathrm{dom}(f) = \mathbb{N} \text{ and } \mathrm{ran}(f) \subseteq \mathbb{N}\} \text{ and} \\ Y &= \{f : \mathrm{dom}(f) = \mathbb{N} \text{ and } \mathrm{ran}(f) \subseteq \mathbb{N} \text{ and } \forall n\,[f(n) < f(n+1)]\} \end{aligned}$$

and one has to define which function $g(f)$ is for each $f \in X$.

**Homework 2.9**
Use the Theorem of Schröder and Bernstein to show that the following two sets $V$ and $W$ have the same cardinality:

$$\begin{aligned} V &= \{f : \mathrm{dom}(f) = \mathbb{N} \text{ and } \mathrm{ran}(f) = \mathbb{N}\} \text{ and} \\ W &= \mathbb{P}(\mathbb{N}). \end{aligned}$$

**Homework 2.10**
Which of the following sets is a function is a valid pair-function on the natural numbers: Note that a valid pair-function must satisfy for all $a, b, c, d \in \mathbb{N}$ that $\langle a, b\rangle = \langle c, d\rangle$ iff $a = c$ and $b = d$. Furthermore, say which of these functions is a bijection from $\mathbb{N} \times \mathbb{N}$ to $\mathbb{N}$. The candidate functions are the following ones:

1. $a, b \mapsto 2^a \cdot 3^b$;
2. $a, b \mapsto (a + b)^3 + b^3$;
3. $a, b \mapsto (a + b)^2 - (a - b)^2$;
4. $a, b \mapsto (0 + 1 + 2 + \ldots + (a + b)) + a$;
5. $a, b \mapsto 2^{a \cdot b} + b$.

**Homework 2.11**
Consider the set $\mathbb{D} = \{q : q$ is a rational number with $0 \leq q < 1$ such that its denominator is of the form $2^i \cdot 3^j$ for some $i, j \in \mathbb{N}\}$. Construct a bijection from $\mathbb{N}$ to $\mathbb{D}$ explicitly.