Homework 2.1
Cantor’s function $x, y \mapsto (x + y) \cdot (x + y + 1)/2 + y$ is a bijection from $\mathbb{N} \times \mathbb{N}$ onto $\mathbb{N}$.
Construct a bijection from $\mathbb{Z} \times \mathbb{Z}$ onto $\mathbb{Z}$.

Homework 2.2
Prove that there is no set $X$ such that its powerset $\{Y : Y \subseteq X\}$ has five elements.

Homework 2.3
Show that a power set has always more elements than the given set, that is, fill out the missing details at the following proof-sketch. Recall that $\text{Card}(A) \leq \text{Card}(B)$ iff there is a one-one function from $A$ to $B$ and show that $\text{Card}(\mathcal{P}(A)) \not\leq \text{Card}(A)$.

Proof-Sketch: The $\emptyset$ has 0 and $\mathcal{P}(\emptyset)$ has one element, namely $\emptyset$, hence one cannot have a one-one mapping from $\mathcal{P}(\emptyset)$ to $\emptyset$. Now assume that $A$ is not empty and $f : A \to \mathcal{P}(A)$ is a function. Show that there is a set $B \subseteq A$ which is not in the range of $f$. Then consider any function $g : \mathcal{P}(A) \to A$ and prove that this function cannot be one-one, as otherwise a surjective $f$ from $A$ to $\mathcal{P}(A)$ would exist. Hence $\text{Card}(\mathcal{P}(A)) \not\leq \text{Card}(A)$.

Homework 2.4
Use Homework 2.3 to prove that there is no set $X$ such that $\text{Card}(\mathcal{P}(X)) = \aleph_0$. The fact that every set $X$ is either finite or satisfies $\aleph_0 \leq \text{Card}(X)$ can be used in the proof.

Homework 2.5
Determine the cardinality of the set $\{X \subseteq Y : \text{Card}(X) = 3\}$ for each of the following sets $Y$: (a) $Y = \{0, 1\}$; (b) $Y = \{0, 1, 2, 3, 4\}$; (c) $Y = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$; (d) $Y = \mathbb{N}$.

Homework 2.6
Consider the set $\mathbb{D} = \{q : q$ is a rational number with $0 \leq q < 1$ such that its denominator is a power of 2\}. Construct a bijection from $\mathbb{N}$ to $\mathbb{D}$ explicitly.

Homework 2.7
A set $X$ is well-ordered iff every non-empty subset $Y \subseteq X$ contains a minimal element, that is, an element $z \in Y$ with $z < u$ for all $u \in Y - \{z\}$. A set $X$ is strongly well-ordered iff every non-empty subset $Y \subseteq X$ contains a minimal element and a maximal element. Determine for which cardinals $\kappa$ there is a strongly well-ordered set $X$ with $\text{Card}(X) = \kappa$.  

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Homework 2.8
Consider the following sets $X$ and $Y$ and show that they satisfy \( \text{Card}(X) = \text{Card}(Y) \) by constructing explicitly a bijection $g$ from $X$ to $Y$. Here

\[
X = \{ f : \text{dom}(f) = \mathbb{N} \text{ and ran}(f) \subseteq \mathbb{N} \} \quad \text{and} \quad 
Y = \{ f : \text{dom}(f) = \mathbb{N} \text{ and ran}(f) \subseteq \mathbb{N} \text{ and } \forall n [ f(n) < f(n + 1)] \}
\]

and one has to define which function $g(f)$ is for each $f \in X$.

Homework 2.9
Use the Theorem of Schröder and Bernstein to show that the following two sets $V$ and $W$ have the same cardinality:

\[
V = \{ f : \text{dom}(f) = \mathbb{N} \text{ and ran}(f) = \mathbb{N} \} \quad \text{and} \quad 
W = \mathbb{P}(\mathbb{N})
\]

Homework 2.10
Which of the following sets is a function is a valid pair-function on the natural numbers: Note that a valid pair-function must satisfy for all $a, b, c, d \in \mathbb{N}$ that \( \langle a, b \rangle = \langle c, d \rangle \) iff $a = c$ and $b = d$. Furthermore, say which of these functions is a bijection from $\mathbb{N} \times \mathbb{N}$ to $\mathbb{N}$. The candidate functions are the following ones:

1. $a, b \mapsto 2^a \cdot 3^b$;
2. $a, b \mapsto (a + b)^3 + b^3$;
3. $a, b \mapsto (a + b)^2 - (a - b)^2$;
4. $a, b \mapsto (0 + 1 + 2 + \ldots + (a + b)) + a$;
5. $a, b \mapsto 2^{a-b} + b$.
6. $a, b \mapsto 2^{(a+1)-(b+1)} + b$.

Homework 2.11
Consider the set $\mathbb{D} = \{ q : q \text{ is a rational number with } 0 \leq q < 1 \text{ such that its denominator is of the form } 2^i \cdot 3^j \text{ for some } i, j \in \mathbb{N} \}$. Construct a bijection from $\mathbb{N}$ to $\mathbb{D}$ explicitly.