MA 4207 - Mathematical Logic
Course-Webpage `http://www.comp.nus.edu.sg/~fstephan/mathlogicug.html`
Homework

Frank Stephan. Departments of Mathematics and Computer Science,
10 Lower Kent Ridge Road, S17#07-04 and 13 Computing Drive, COM2#03-11,
National University of Singapore, Singapore 119076.
Email fstephan@comp.nus.edu.sg
Telephone office 65162759 and 65164246
Office hours Thursday 14.00-15.00h at Mathematics S17#07-04

Homework 2.1
Cantor's function $x, y \mapsto (x + y) \cdot (x + y + 1)/2 + y$ is a bijection from $\mathbb{N} \times \mathbb{N}$ onto $\mathbb{N}$. Construct a bijection from $\mathbb{Z} \times \mathbb{Z}$ onto $\mathbb{Z}$.

Homework 2.2
Prove that there is no set X such that its powerset $\{Y : Y \subseteq X\}$ has five elements.

Homework 2.3
Show that a power set has always more elements than the given set, that is, fill out the missing details at the following proof-sketch. Recall that $\text{Card}(A) \leq \text{Card}(B)$ iff there is a one-one function from A to B and show that $\text{Card}(\mathbb{P}(A)) \not\leq \text{Card}(A)$.

Proof-Sketch: The $\emptyset$ has 0 and $\mathbb{P}(\emptyset)$ has one element, namely $\emptyset$, hence one cannot have a one-one mapping from $\mathbb{P}(\emptyset)$ to $\emptyset$. Now assume that A is not empty and $f : A \to \mathbb{P}(A)$ is a function. Show that there is a set $B \subseteq A$ which is not in the range of f. Then consider any function $g : \mathbb{P}(A) \to A$ and prove that this function cannot be one-one, as otherwise a surjective f from A to $\mathbb{P}(A)$ would exist. Hence $\text{Card}(\mathbb{P}(A)) \not\leq \text{Card}(A)$.

Homework 2.4
Use Homework 2.3 to prove that there is no set X such that $\text{Card}(\mathbb{P}(X)) = \aleph_0$. The fact that every set X is either finite or satisfies $\aleph_0 \leq \text{Card}(X)$ can be used in the proof.

Homework 2.5
Determine the cardinality of the set $\{X \subseteq Y : \text{Card}(X) = 3\}$ for each of the following sets Y: (a) $Y = \{0, 1\}$; (b) $Y = \{0, 1, 2, 3, 4\}$; (c) $Y = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$; (d) $Y = \mathbb{N}$.

Homework 2.6
Consider the set $\mathbb{D} = \{q : q$ is a rational number with $0 \leq q < 1$ such that its denominator is a power of $2\}$. Construct a bijection from $\mathbb{N}$ to $\mathbb{D}$ explicitly.

Homework 2.7
A set X is well-ordered iff every non-empty subset $Y \subseteq X$ contains a minimal element, that is, an element $z \in Y$ with $z < u$ for all $u \in Y - \{z\}$. A set X is strongly well-ordered iff every non-empty subset $Y \subseteq X$ contains a minimal element and a maximal element. Determine for which cardinals κ there is a strongly well-ordered set X with $\text{Card}(X) = \kappa$.

Homework 2.8
Consider the following sets X and Y and show that they satisfy $\mathrm{Card}(X) = \mathrm{Card}(Y)$ by constructing explicitly a bijection g from X to Y. Here

$$\begin{aligned} X &= \{f : \mathrm{dom}(f) = \mathbb{N} \text{ and } \mathrm{ran}(f) \subseteq \mathbb{N}\} \text{ and} \\ Y &= \{f : \mathrm{dom}(f) = \mathbb{N} \text{ and } \mathrm{ran}(f) \subseteq \mathbb{N} \text{ and } \forall n\, [f(n) < f(n+1)]\} \end{aligned}$$

and one has to define which function $g(f)$ is for each $f \in X$.

Homework 2.9
Use the Theorem of Schröder and Bernstein to show that the following two sets V and W have the same cardinality:

$$\begin{aligned} V &= \{f : \mathrm{dom}(f) = \mathbb{N} \text{ and } \mathrm{ran}(f) = \mathbb{N}\} \text{ and} \\ W &= \mathbb{P}(\mathbb{N}). \end{aligned}$$

Homework 2.10
Which of the following sets is a function is a valid pair-function on the natural numbers: Note that a valid pair-function must satisfy for all $a, b, c, d \in \mathbb{N}$ that $\langle a, b\rangle = \langle c, d\rangle$ iff $a = c$ and $b = d$. Furthermore, say which of these functions is a bijection from $\mathbb{N} \times \mathbb{N}$ to $\mathbb{N}$. The candidate functions are the following ones:

1. $a, b \mapsto 2^a \cdot 3^b$;
2. $a, b \mapsto (a + b)^3 + b^3$;
3. $a, b \mapsto (a + b)^2 - (a - b)^2$;
4. $a, b \mapsto (0 + 1 + 2 + \ldots + (a + b)) + a$;
5. $a, b \mapsto 2^{a \cdot b} + b$.
6. $a, b \mapsto 2^{(a+1)\cdot(b+1)} + b$.

Homework 2.11
Consider the set $\mathbb{D} = \{q : q$ is a rational number with $0 \leq q < 1$ such that its denominator is of the form $2^i \cdot 3^j$ for some $i, j \in \mathbb{N}\}$. Construct a bijection from $\mathbb{N}$ to $\mathbb{D}$ explicitly.