MA 4207 - Mathematical Logic
Course-Webpage `http://www.comp.nus.edu.sg/~fstephan/mathlogicug.html`
Homework

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Homework 4.1
Let $atom(\phi)$ be the set of atoms used in ϕ, so $atom(((A_1 \vee A_2) \wedge A_2)) = \{A_1, A_2\}$ and $atom((0 \vee 1)) = \emptyset$. Let WFF be the set of well-formed formulas. Let

$$C_1 = \{\phi \in WFF : \forall v\,[\text{if } v(A) = 1 \text{ for some } A \in atom(\phi) \text{ then } \overline{v}(\phi) = 1]\}.$$

For which of the connectives $\neg, \wedge, \vee, \rightarrow, \leftrightarrow, \oplus$ is C_1 closed under the connective? Here one says that C is closed under the connective $\oplus$ if all formulas $\phi, \psi \in C$ satisfy that $(\phi \oplus \psi) \in C$. Similarly for other connectives.

Homework 4.2
Let $atom(\phi)$ and WFF be defined as in Homework 4.1. Let

$$C_2 = \{\phi \in WFF : \forall v\,[\text{if } v(A) = 0 \text{ for at most one } A \in atom(\phi) \text{ then } \overline{v}(\phi) = 1]\}.$$

For which of the connectives $\neg, \wedge, \vee, \rightarrow, \leftrightarrow, \oplus$ is C_2 closed under the connective?

Homework 4.3
Let $C_3 = \{\phi \in WFF :$ every $A \in atom(\phi)$ occurs in ϕ exactly once and $0, 1$ do not occur in $\phi\}$. Prove by induction that C_3 does not contain any tautology and also not contain any antitautology. Here a tautology is a formula which is always true (independent of the choice of the truth-values of the atoms) and an antitautology is a formula which is always false.

Homework 4.4
Define on WFF by recursion the functions $atom$ ($\phi \mapsto atom(\phi)$) and $maxatom$ ($\phi \mapsto \max\{k : A_k \in atom(\phi)\}$), where the atoms $A_1, A_2, \ldots$ can be used and $\max \emptyset = 0$. So $maxatom((A_1 \vee (A_4 \wedge A_5))) = 5$ and $maxatom((0 \vee 1)) = 0$.

Homework 4.5
Define on WFF by recursion the function $numcon(\phi)$ as the number of connectives $\wedge, \vee, \rightarrow, \leftrightarrow, \oplus$ occurring in ϕ. Furthermore define the function $numneg(\phi)$ as the number of negations occurring in ϕ. Determine the best-possible constants c, m, n such that

$$|\phi| \leq c \cdot numcon(\phi) + n \cdot numneg(\phi) + m$$

for all WFF ϕ.

Homework 4.6
Let $C_6 = \{\phi \in WFF : \text{every } A \in atom(\phi) \text{ occurs in } \phi \text{ exactly once}\}$; note that formulas in C_6 might have occurrences of the constants 0 and 1. Define by recursion a function F from C_6 into the rational numbers between 0 and 1 which returns for each formula $\phi \in C_6$ the truth-probability $n/2^m$ where n is the number of rows in the truth-table of ϕ evaluated to 1 and m is the number of atoms used in the formula so that 2^m is the overall number of rows in the truth-table of ϕ. For example, $F(1) = 1$, $F((A_1 \oplus (A_2 \vee A_3))) = 1/2$ and $F(((A_2 \vee A_5) \wedge (A_3 \vee 0))) = 3/8$.

Homework 4.7
Let $C_7 = \{\phi \in WFF : \phi$ can use the constants $0, 1$ and the only connectives in ϕ are $\wedge$ and $\vee\}$. Prove by induction that a formula $\phi \in C_7$ is a tautology iff $\overline{v}(\phi) = 1$ for the truth-assignment v with $v(A_k) = 0$ for all k.

Homework 4.8
Let $C_8 = \{\phi \in WFF : \phi$ can use the constants $0, 1$ and the only connectives in ϕ are $\wedge$ and $\vee\}$. Prove by induction that a formula $\phi \in C_8$ is an antitautology iff $\overline{v}(\phi) = 0$ for the truth-assignment v with $v(A_k) = 1$ for all k.

Homework 4.9
Let U be a finite set of atoms and $C_9 = \{\phi \in WFF : atom(\phi) \subseteq U\}$. Prove that there is a finite set F of formulas such that for every formula $\phi \in C_9$ there is a $\psi \in F$ with $(\psi \leftrightarrow \phi)$ being a tautology.

Homework 4.10
The following formulas have brackets omitted according to the rule that the binding strengths of the connectives is ordered as $\neg, \wedge, \vee, \oplus, \rightarrow, \leftrightarrow$. Insert back the needed brackets for getting a member of WFF.

1. $A_1 \wedge \neg A_2 \vee A_3 \rightarrow A_4 \wedge \neg A_5$;
2. $A_1 \vee \neg A_2 \wedge \neg A_3 \leftrightarrow A_4 \rightarrow A_5$;
3. $\neg\neg A_1 \vee \neg A_2$.

Homework 4.11
Is there a formula using the connectives $\oplus$ and $\neg$ but no other connectives where the value of the formula depends on the placement of brackets?

Homework 4.12
Let $v(A_1) = 1$, $v(A_2) = 1$, $v(A_3) = 0$. The below formulas are given in Polish notation. Write them as WFF and evaluate them according to v:

1. $\neg \leftrightarrow \oplus A_1 A_2 A_3$;
2. $\wedge \vee \neg \vee A_1 A_2 A_3 A_1$;
3. $\oplus \wedge A_1 A_2 \wedge A_2 A_3$.

Homework 4.13
Write the following formula in Polish notation: $\neg((A_1 \vee \neg A_2) \wedge (A_2 \vee \neg A_3) \wedge (A_3 \vee \neg A_1))$.

Homework 4.14
Consider the formula ϕ in 9 atoms $A_1, \ldots, A_9$ and consider the set S of all $(A_i \oplus A_j \oplus A_k)$ with $A_i \in \{A_1, A_2, A_3\}$, $A_j \in \{A_4, A_5, A_6\}$ and $A_k \in \{A_7, A_8, A_9\}$. Now let ϕ_1 be the conjunction of all formulas in S; that is, if $S = \{\psi_1, \psi_2, \ldots, \psi_n\}$ then $\phi_1 = \psi_1 \wedge \psi_2 \wedge \ldots \wedge \psi_n$ when the brackets are omitted. Determine how many assignments of these atoms make the formula true and how many make it false.

Homework 4.15
Let ϕ_2 to be taken the disjunction of the formulas in set S from Homework 4.14. Determine how many assignments of these atoms make the formula true and how many make it false.

Homework 4.16
The following formulas have brackets omitted according to the rule that the binding strengths of the connectives is ordered as $\neg, \wedge, \vee, \oplus, \rightarrow, \leftrightarrow$. Insert back the needed brackets for getting a member of WFF and evaluate the formulas for the truth-assignment ν which sets all atoms to 1.

1. $A_1 \vee \neg A_2 \wedge A_3 \leftrightarrow A_4 \rightarrow \neg A_5 \vee A_3 \wedge A_6 \oplus A_7$;
2. $A_1 \rightarrow A_2 \rightarrow A_3 \rightarrow A_4 \rightarrow \neg A_5$;
3. $\neg\neg\neg A_1 \oplus A_2 \oplus A_3$.