MA 4207 - Mathematical Logic
Course-Webpage `http://www.comp.nus.edu.sg/~fstephan/mathlogicug.html`
Homework

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Homework 6.1
Which of the following statements are true? Prove your answers.
(a) $\{\alpha, \beta\} \models c \vee d \Leftrightarrow \{\alpha, \beta\} \models c$ or $\{\alpha, \beta\} \models d$.
(b) $\{\alpha, \beta\} \models c \wedge d \Leftrightarrow \{\alpha, \beta\} \models c$ and $\{\alpha, \beta\} \models d$.
(c) $\{\alpha, \beta\} \models \alpha \oplus \beta \Leftrightarrow \alpha \wedge \beta$ is not satisfiable.
Here a formula α is satisfiable iff there is a choice of truth-values of the atoms such that α becomes true.

Homework 6.2
Which of the following statements are true? Prove your answers.
(a) $S \models \alpha \Leftrightarrow S \cup \{\alpha\}$ is satisfiable.
(b) $S \models \alpha \Leftrightarrow S \cup \{\neg\alpha\}$ is not satisfiable.
(c) $S \models \alpha \rightarrow \beta \Leftrightarrow S \cup \{\neg\alpha\} \models \neg\beta$.
Here a set S of formulas is satisfiable iff there is a choice of truth-values of the atoms such that all formulas in S are true.

Homework 6.3
Make an infinite set S of formulas such that every subset of two formulas is satisfiable but no subset of three or more formulas is.

Homework 6.4
Is the set $\{\leftrightarrow, \neg, \oplus, 0, 1\}$ of connectives and constants complete? Do the subsets $\{\leftrightarrow, 1\}$ and $\{\leftrightarrow, 0\}$ have the same expressive power or less expressive power than $\{\leftrightarrow, \neg, \oplus, 0, 1\}$?

Homework 6.5
Make a formula in A_1, A_2, A_3, A_4 with as few of the connectives $\wedge$ and $\vee$ as possible, but which might use as many $\neg$ as needed such that the following constraints are satisfied: If none or all four of the atoms are 1 then the output is 0 and if one or three of the atoms are 1 then the output is 1; there is no requirment on what happens if exactly two atoms are 1. Use Enderton's Square Method.

Homework 6.6
Make a formula in A_1, A_2, A_3, A_4 with as few of the connectives $\wedge$ and $\vee$ as possible, but which might use as many $\neg$ as needed such that the following constraints are satisfied: If none or three of the atoms are 1 then the output is 0 and if one or all

four of the atoms are 1 then the output is 1; there is no requirment on what happens if exactly two atoms are 1. Use Enderton's Square Method.

Homework 6.7
For switching circuits based on relays and with the possibility to use both normal and negated inputs, construct a circuit which uses as few input-invokations as possible in order to compute the majority-function in three variables.

Homework 6.8
Consider the three-valued fuzzy logic with truth-values from $Q = \{0, 1/2, 1\}$. Can the set of $\{\wedge, \vee, \neg, \oplus, \leftrightarrow, \rightarrow\}$ plus the three truth-values be used to generate all functions from $Q^2 \rightarrow Q$?

Homework 6.9
Consider fuzzy logic with truth-values from some finite Q satisfying the constraints from Chapter 1.5. Find a set containing only three connectives which is, together with the truth-values, as powerful as the set $\{\wedge, \vee, \neg, \oplus, \leftrightarrow, \rightarrow\}$ with respect to the ability to generate functions from $Q^2 \rightarrow Q$.

Homework 6.10
Consider fuzzy logic with $Q = \{r \in \mathbb{R} : 0 \leq r \leq 1\}$. Provide some examples of functions from Q to Q which are not equal to B^1_α for some α generated by rational truth-values and the connectives of fuzzy logic in Chapter 1.5.

Homework 6.11-6.13
Corollary 17A says that if $S \models \alpha$ then there is a finite subset S' of S such that $S' \models \alpha$. This proof does not directly translate to fuzzy logic and indeed, if one defines $S \models \alpha$ in fuzzy logic in the wrong way, then it is false. For the following homeworks, consider $Q = \{r \in \mathbb{R} : 0 \leq r \leq 1\}$ and $S = \{q \rightarrow A_1 : q \in \mathbb{Q} \text{ and } 0 \leq q < 1\}$ and $\alpha = A_1$.

Homework 6.11
Assume that one defines $S \models \alpha$ as "All Q-valued truth-assignments ν satisfy that if $\overline{\nu}(\beta) = 1$ for all $\beta \in S$ then $\overline{\nu}(\alpha) = 1$" and show that then $S \models \alpha$ but no finite subset S' of S satisfies $S' \models \alpha$.

Homework 6.12
Assume that one defines $S \models \alpha$ as "All Q-valued truth-assignments ν and all $\varepsilon > 0$ satisfy that there is $\beta \in S \cup \{1\}$ with $\overline{\nu}(\beta) \leq \overline{\nu}(\alpha) + \varepsilon$" and show that then $S \models \alpha$ but no finite subset S' of S satisfies $S' \models \alpha$.

Homework 6.13
Assume that one defines $S \models \alpha$ as "All Q-valued truth-assignments ν and all $q \in Q$ satisfy that if $\overline{\nu}(\beta) \geq q$ for all $\beta \in S$ then $\overline{\nu}(\alpha) \geq q$" and show that then $S \models \alpha$ but no finite subset S' of S satisfies $S' \models \alpha$.