Recall that a set is recursively enumerable iff it is empty or is the range of a function computed by an effective procedure (also called recursive function). Consider now three $X, Y, Z$ be effectively enumerable sets which all contain 0 and which are the ranges of functions $F_X, F_Y, F_Z$ which are given by effective procedures. Make recursive functions $G, H$ such that the range of $G$ is $X \cup Y \cup Z$ and the range of $H$ is $(X \cap Y) \cup (X \cap Z) \cup (Y \cap Z) = \{ u : u \text{ is in at least two of the sets } X, Y, Z \}$.

Homework 7.2
Prove that the following set $S$ is decidable: $S$ is the set of all $n$ such that there are infinitely many natural numbers $m$ for which both $m$ and $m + n$ are powers of 2.

Homework 7.3
Prove that the following set $S$ is decidable: $S$ contains all numbers $x$ for which there are infinitely many pairs $y, z$ of prime numbers satisfying that $y < z \leq y + x$.

Homework 7.4
A binary tree $T$ is a set of binary strings such that whenever $\sigma \tau \in T$ then $\sigma \in T$ (where $\sigma \tau$ is the concatenation of $\sigma$ and $\tau$). König’s Lemma says that every infinite binary tree contains an infinite branch. Now let $A_1, A_2, \ldots$ be the atoms and let $S = \{ \alpha_1, \alpha_2, \ldots \}$ be a set of formulas. Now let $T$ be a binary tree which on level $n$ contains all those $\sigma \in \{0, 1\}^n$ which satisfy for all formulas $\beta \in \{ \alpha_1, \alpha_2, \ldots, \alpha_n \}$, if no atom $A_k$ with $k > n$ occurs in $\beta$ then every $v$ with $v(A_k) = \sigma(k)$ makes $\beta$ true. Prove the following: If $T$ is infinite then $T$ has an infinite branch and each infinite branch defines a $v$ with $v \models S$; if $T$ is finite then there is a first level $n$ on which $T$ has no nodes and $\{ \alpha_1, \alpha_2, \ldots, \alpha_n \}$ is not satisfiable.

Homework 7.5
Let $S = \{ \alpha : \alpha$ is a well-formed formula and $\overline{\nu}(\alpha) = 1$ iff the majority of the atoms $A_k \in \text{atom}(\alpha)$ are 1 $\}$. Prove that $S$ is decidable.

Homework 7.6
Let $\nu$ be computed by an effective procedure mapping each $k \in \mathbb{N}$ to the truth-value assigned to atom $A_k$. Let $S = \{ \alpha \in WFF : \overline{\nu}(\alpha) = 1 \}$. Which of the following options is correct? (a) $S$ is decidable; (b) $S$ is recursively enumerable but not decidable; (c) $S$ is not
Homework 7.7
Assume that you know that addition, subtraction and multiplication are effectively computable. Use now recursion in one variable to show that (a) the integer division $n, m \mapsto \max\{k : k \cdot m \leq n\}$ and (b) $n \mapsto \binom{2n}{n}$ are effectively computable functions. Note that the recursion can use case-distinctions; for example, the inductive definition of the remainder $f(a, b)$ of $a$ by $b$ is $f(0, b) = 0$ and if $f(a, b) + 1 < b$ then $f(a + 1, b) = f(a, b) + 1$ else $f(a + 1, b) = 0$.

Homework 7.8
Prove that if $S$ is a satisfiable set of formulas then $WFF - S$ is not a satisfiable set of formulas.

Homework 7.9
Assume that $S_1, S_2, S_3$ are satisfiable sets of formulas. What about the set $T = (S_1 \cup S_2) \cap (S_1 \cup S_3) \cap (S_2 \cup S_3)$? Prove that $T$ is satisfiable or give an example of $S_1, S_2, S_3$ where the resulting $T$ is not satisfiable.

Homework 7.10
Let $S = \{\alpha \in WFF : \nu(\alpha) = 1\}$ for some $\nu$ and $T = \{\alpha : (\alpha \lor A_1), (\alpha \lor (\neg A_1)) \in S\}$. Is $T$ satisfiable? Is $S = T$?

Homework 7.11
Call a set $S$ of formulas almost-zero-satisfiable (azs) iff there is a $\nu$ with $\nu(A_k) = 0$ for almost all atoms and $\nu(\alpha) = 1$ for all $\alpha \in S$. Does the notion “azs” satisfy the compactness theorem? That is, for any infinite set $S \subseteq WFF$, if every finite subset is almost-zero-satisfiable, is then $S$ itself also almost-zero-satisfiable?

Homework 7.12
Are there infinite sets $S, T$ of wff such that every finite subset $T'$ of $T$ there is a finite subset $S'$ of $S$ such that $S' \models \alpha$ for all $\alpha \in T'$ but it does not hold that $S \models T$, that is, there is some $\nu$ which is true on all members of $S$ but not all members of $T$.

Homework 7.13
Assume that there are infinitely many logical atoms. Is there a set $S$ of formulas such that for all $\nu$ mapping atoms to $\{0, 1\}$, $\nu \models S$ iff there are exactly three atoms $A, B, C$ with $\nu(A) = 1, \nu(B) = 1, \nu(C) = 1$?