MA 4207 - Mathematical Logic
Course-Webpage `http://www.comp.nus.edu.sg/~fstephan/mathlogicug.html`
Homework

Frank Stephan. Departments of Mathematics and Computer Science,
10 Lower Kent Ridge Road, S17#07-04 and 13 Computing Drive, COM2#03-11,
National University of Singapore, Singapore 119076.
Email fstephan@comp.nus.edu.sg
Telephone office 65162759 and 65164246
Office hours Thursday 14.00-15.00h at Mathematics S17#07-04

Homework 8.1
Let $Ap(x)$ say "x is an apple", $Ba(x)$ say "x is a banana", $Cb(x)$ say "x is a cranberry" and $Cu(x)$ say "x is a currant". Furthermore, let $Ye(x)$ say "x is yellow", $Re(x)$ say "x is red" and $Bl(x)$ say "x is black". Now translate the following English sentences into logic:

1. There are yellow apples and red apples.
2. All bananas are yellow.
3. Cranberries are always red.
4. There are red currants and black currants and every currant has one of these two colours.

Homework 8.2
Given the notation from homework 8.2, translate the following formulas into normal English language sentences:

$$\begin{aligned} \forall x \quad & [Ap(x) \rightarrow \neg Ba(x)]; \\ \exists x \quad & [Bl(x) \wedge \neg Ap(x) \wedge \neg Ba(x)]; \\ \forall x \, \forall y \quad & [Re(x) \wedge Bl(y) \rightarrow x \neq y]; \\ \forall x \, \exists y \quad & [(Cu(x) \wedge Re(x)) \rightarrow (Cu(y) \wedge Bl(y))]. \end{aligned}$$

Homework 8.3
Assume that there is a set X of five fruits satisfying the following formulas.

$$\begin{aligned} \forall x \quad & [Ap(x) \vee Ba(x) \vee Cu(x)]; \\ \forall x \quad & [(\neg Ap(x) \wedge \neg Ba(x)) \vee (\neg Ap(x) \wedge \neg Cu(x)) \vee (\neg Ba(x) \wedge \neg Cu(x))]; \\ \forall x \quad & [Bl(x) \vee Re(x) \vee Ye(x)]; \\ \forall x \quad & [(\neg Bl(x) \wedge \neg Re(x)) \vee (\neg Bl(x) \wedge \neg Ye(x)) \vee (\neg Re(x) \wedge \neg Ye(x))]; \\ \forall x \quad & [Ap(x) \rightarrow \neg Bl(x)]; \\ \forall x \quad & [Ba(x) \rightarrow Ye(x)]; \\ \forall x \quad & [Cu(x) \rightarrow \neg Ye(x)]; \\ \exists u \, \exists v \, \exists w \, \exists x \, \exists y \quad & [Re(u) \wedge v \neq w \wedge Ye(v) \wedge Ye(w) \wedge x \neq y \wedge Bl(x) \wedge Bl(y)]. \end{aligned}$$

Calculate the number of models (up to isomorphism) which satisfy these formulas with five elements.

Homework 8.4
Use the formulas from Homework 8.3, but assume that X has 6 elements. Calculate the number of models (up to isomorphism) which satisfy these formulas with six elements.

Homework 8.5
Use the formulas from Homework 8.3, but assume that X has at most 4 elements. Calculate the number of models (up to isomorphism) which satisfy these formulas with up to four elements.

Homework 8.6
Assume that equality is in the logical language, but no predicate or function. Make a set S of formulas which says that the number of elements of a structure satisfying S is either a prime number or infinite. This set S is infinite.

Homework 8.7
Assume that a structure X with one function symbol f satisfies

$$\forall x\,[f(x) \neq x \wedge f(f(x)) = x].$$

What can be said about the number of elements in the base set X?

Homework 8.8
Make a formula using the language of natural numbers with addition and order which says that there are infinitely many numbers which are not multiples of any of 2, 3 and 5. This formula should not use multiplication.

Homework 8.9
Consider the structure $(\mathbb{N}, +, -, \cdot, <, =, 0, 1, 2, \ldots)$ and the corresponding first-order logical language of arithmetic with constants for every natural number. Make formulas which express the following:

1. Each number is either 0 or 1 or the multiple of a prime number;
2. There are infinitely many prime numbers of the form $5n + 1$.

Homework 8.10
Consider the structure $(\mathbb{N}, +, -, \cdot, <, =, 0, 1, 2, \ldots)$ and the corresponding first-order logical language of arithmetic with constants for every natural number. Make formulas which express the following:

1. Every even number other than 0 and 2 is the sum of two prime numbers;
2. There are infinitely many numbers x such that $x - 1$ and $x + 1$ are both prime numbers.

Homework 8.11

Consider the structure $(\mathbb{Z}, +, -, \cdot, <, =, 0, -1, 1, -2, 2, \ldots)$ and the corresponding first-order logical language of arithmetic with constants for every integer. Make formulas which express the following:

1. The number 23 is not the sum of three squares;
2. A number is the sum of four squares if it is greater or equal 0.

Homework 8.12

For first-order logic, assume that the logical language has only equality and variables and quantifiers and the logical connectives. The formula $\exists x, y, z\, [x \neq y \wedge x \neq z \wedge y \neq z]$ can only be satisfied by a structure with at least three elements. Is there, in this logical language, a formula α which can only be satisfied by structures with infinitely many elements? Is there a set S of formulas such that S is only satisfied by structures with infinitely many elements?

Homework 8.13

Let $(F, +, -, \cdot, f, =, 0, 1, 2)$ be the finite field with the three elements $0, 1, 2$ and let $f : F \to F$ be any function. Which of the following statements are true for this structure (independently of how f is chosen)?

1. $\forall x, y\, [(x + y) \cdot (x + y) = (x \cdot x) + (y \cdot y) - (x \cdot y)]$;
2. $\forall x, y\, [(x + y) \cdot (x + y) \cdot (x + y) = (x \cdot x \cdot x) + (y \cdot y \cdot y)]$;
3. $\forall x, y\, [(x + y) \cdot (x + y) \cdot (x + y) \cdot (x + y) = (x \cdot x \cdot x \cdot x) + (y \cdot y \cdot y \cdot y)]$;
4. $\exists a, b, c\, \forall x\, [f(x) = a \cdot x \cdot (x - 1) + b \cdot x \cdot (x - 2) + c \cdot (x - 1) \cdot (x - 2)]$;
5. $\forall x\, [x \cdot x \cdot x \neq 2]$.