MA 4207 - Mathematical Logic
Course-Webpage `http://www.comp.nus.edu.sg/~fstephan/mathlogicug.html`
Homework

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Homework 9.1
Let $(A, +, \cdot)$ and $(B, +, \cdot)$ be the remainder rings modulo a and b, respectively, $a, b \in \{2, 3, 4, 5, 6\}$. For which a, b is there a homomorphism f from $(A, +, \cdot)$ to $(B, +, \cdot)$ such that any two terms t_1, t_2 satisfy $(A, +, \cdot), s \models t_1 = t_2$ iff $(B, +, \cdot), s' \models f(t_1) = f(t_2)$, where $s'(v_k) = f(s(v_k))$ for all variables v_k.

Homework 9.2
Choose values for a, b from Homework 9.1 and a formula ϕ and a function f such that f is a homomorphism and the formula ϕ is true in $(A, +, \cdot)$ but not in $(B, +, \cdot)$.

Homework 9.3
Consider the model $(\{0, 1, \ldots, 9\}, +, \cdot)$ with addition and multiplication modulo 10, so $5 + 7 = 2$ and $5 \cdot 7 = 5$. Which are the sets defined by the following formulas:

1. $x \in A \Leftrightarrow \exists y\, [x = y \cdot y]$;
2. $x \in B \Leftrightarrow \forall y\, [x \cdot y = 0 \vee x \cdot y = 3 \vee x \cdot y = 5]$;
3. $x \in C \Leftrightarrow \forall y\, [x \neq y \cdot y \cdot y \cdot y]$.

Homework 9.4
Is the set $\{2, 4, 6, 8\}$ definable in the model of arithmetic modulo 10? Here the formula can use the operations $+, \cdot$ and the constants $0, 1$ and equality $=$ and connectives and quantifiers.

Homework 9.5
Is every function in the model $\{0, 1, \ldots, 9\}$ with addition and multiplication and all constants explicitly definable by a term? If so, give a proof; if not, explain why.

Homework 9.6
Let $\mathbb{Z} \cdot \{i\} + \mathbb{Z}$ be the set of all complex integer numbers. Show that this set together with $+$ and $\cdot$ is a ring. Prove that the basis element i is not definable by using an isomorphism which maps i to some other element.

Homework 9.7
Recall that a structure $(A, \circ, e)$ is a group iff it satisfies $\forall x, y, z\, [x \circ (y \circ z) = (x \circ y) \circ z]$, $\forall x\, [x \circ e = x \wedge e \circ x = x]$, $\forall x \exists y\, [x \circ y = e \wedge y \circ x = e]$.
Write down formally the axioms for an Abelian group, a ring with 1 and a commutative

ring with 1, respectively.

Homework 9.8
Assume that $(A, +, \cdot, 0, 1)$ is a finite ring with $0 \neq 1$. Consider the formulas

$$\begin{aligned} x = 0 &\Leftrightarrow \forall y\,[x + y = y] \text{ and} \\ x = 1 &\Leftrightarrow \forall y\,[x \cdot y = y \wedge y \cdot x = y]. \end{aligned}$$

Are then all members of A definable with formulas like this? If yes then prove how this is done else provide a finite ring where some elements are not definable.

Homework 9.9
Let $(\mathbb{R}, +, \cdot, <, 0, 1)$ be the ordered field of the real numbers with the constants 0 and 1. Prove that all rational numbers and all real roots of polynomials are definable. Provide then examples of formulas $\phi_1, \phi_2, \phi_3, \phi_4$ such that x_k is the unique element satisfying ϕ_k where the formulas ϕ_k say the following:

1. $x_1 = 2/3$;
2. x_2 is the positive square-root of 3;
3. x_3 is the largest number satisfying $x_3^{10} - 4x_3^5 + 2 = 0$;
4. x_4 is the smallest number satisfying $3x_4^6 - 6x_4^4 + 3x_4^2 = 0$.

Homework 9.10
Consider a structure $(A, f, 0, 1, =)$ with $0, 1 \in A$ being constants and f a function from A to A. Make three formulas in the language of this structure which express the following conditions:

1. The first formula says that f has the range $\{0, 1\}$;
2. The second formula says that f is the inverse of itself;
3. The third formula says that every value in the range of f is the image of exactly two values.

Homework 9.11
Let $(A, P^A), (B, P^B)$ be two structures with $A = \{0\}$ and $B = \{1, 2\}$. choose the predicate P^B such that there is no strong homomorphism from B to A (independently of what P^A is) while there is for each possible choice of P^A a strong homomorphism from (A, P^A) to (B, P^B).

Homework 9.12
Let $(\mathbb{Z}, Succ, Even)$ be a structure with $Even(x)$ being true iff x is even and $Succ$ being the successor function. Let f be a function from the structure to itself. Prove that if f is a homomorphism then f is a strong homomorphism.

Homework 9.13
Consider the structure $(\mathbb{Z}, Neigh, Even)$ where $Even(x)$ is true iff x is even and

$Neigh(x, y)$ is true iff $x = y + 1$ or $x = y - 1$. Construct a function g from $\mathbb{Z}$ to itself which is a homomorphism but not a strong homomorphism.

Homework 9.14
Assume that $(A, +, a, b, c, d)$ is an n-dimensional vector space for some n over the field $(\{0, 1, 2\}, +, \cdot)$ with three elements; here for the skalar multiplication, $x \cdot 0 = x + x + x$, $x \cdot 1 = x$ and $x \cdot 2 = x + x$, so that the multiplication with each fixed skalar is definable. Find the largest dimension n so that all elements in the vector space $(A, +, a, b, c, d)$ are definable when one chooses the right values for a, b, c, d and explain how the formulas to define the elements look like; note that an isomorphism of the structure itself has to map a to a, b to b, c to c and d to d.

Homework 9.15
Assume that a structure $(X, +)$ satisfies the below axioms:

1. $\forall x\, \forall y\, \forall z\, [x + (y + z) = (x + y) + z]$;
2. $\forall x\, \forall y\, [x + y = y + x]$;
3. $\forall x\, \forall y\, [x + x = y + y]$;
4. $\forall x\, [x + x + x = x]$.

Assume furthermore, that the structure has four elements $0, a, b, c$ and that 0 is the element with $0 = x + x$ for all $x \in X$. Prove (informally, not in the deductive calculus) that the structure satisfies $a + b = c$ and show that a, b, c are not definable by constructing an isomorphism h with $h(a) \neq a$, $h(b) \neq b$ and $h(c) \neq c$.