MA 4207 - Mathematical Logic
Course-Webpage `http://www.comp.nus.edu.sg/~fstephan/mathlogicug.html`
Homework

Frank Stephan. Departments of Mathematics and Computer Science,
10 Lower Kent Ridge Road, S17#07-04 and 13 Computing Drive, COM2#03-11,
National University of Singapore, Singapore 119076.
Email fstephan@comp.nus.edu.sg
Telephone office 65162759 and 65164246
Office hours Thursday 14.00-15.00h at Mathematics S17#07-04

Homework 10.1
This homework considers undirected graphs without self-loops. Consider the following two graphs:

```
(a) 0 - 1 - 2            (b) 0  1 - 2
        | /                     | /
        3                       3
```

(a) Prove that in graph (a) nodes 0 and 1 are definable and nodes 2 and 3 are not.
(b) Prove that in graph (b) the node 0 is definable and nodes 1, 2 and 3 are not.
(c) For which n is there a graph of 6 nodes such that exactly n out of these 6 nodes are definable?

Homework 10.2
Let the logical language contain a function symbol f for a function with one input. Show that Λ proves the formulas

$$\neg(f(x) = f(y)) \to \neg(f(y) = f(x)),\ f(x) = f(y) \to (f(y) = f(z) \to f(x) = f(z))$$

which is similar to some proofs in the lecture notes.

Homework 10.3
For the following formulas α and terms t, either write what α^z_t is or write that a substitution is not permitted. The formulas are $\exists x\,[\neg(x = z{+}1)]$, $\forall z\,[x = z]$, $f(x{\cdot}z) = f(0)$ and the terms are x, 0, $z + z$. Do not forget to make brackets where needed.

Homework 10.4
For the following formulas α and terms t, either write what α^z_t is or write that a substitution is not permitted. The formulas are $\exists x \forall y\,[x = y \cdot z]$, $\forall x\,\exists y\,[z = x + y]$, $\forall u\,[z \cdot z + 1 \neq u \cdot u + 2]$ and the terms are $x + y$, 0, $v \cdot w$. Do not forget to make brackets where needed.

Homework 10.5
Use the Deduction Theorem to show the following:
If $\Gamma \vdash \alpha \to \beta \to \gamma \to \delta$ then $\Gamma \vdash\ \gamma \to \alpha \to \beta \to \delta$.
Which other interchanges of $\alpha, \beta, \gamma, \delta$ are permitted and which not?

Homework 10.6
Prove the statement from Homework 10.5 using only tautologies and modus ponens.

Homework 10.7
Let the logical language have a predicate P and constant c. Prove formally that

$$\{\forall x\,\forall y\,[P(x) \to P(y)]\} \vdash P(c) \to \forall y\,[P(y)]$$

using the axioms of Λ, the Deduction Theorem and the Generalisation Theorem.

Homework 10.8
Let $(A, +, 0)$ be a structure with constant 0 and binary operation $+$. Make a formal proof for

$$\{\forall x\,[x + x = 0]\} \vdash \forall x\,[(x + x) + (x + x) = 0]$$

using axioms from Λ and the Generalisation Theorem.

Homework 10.9
Let $(A, +, 0)$ be a structure with constant 0 and binary operation $+$. Make a formal proof for

$$\{\forall x\,\forall y\,[x + y = y + x]\} \vdash \forall u\,[u + (u + u) = (u + u) + u]$$

using the axioms of Λ and the Generalisation Theorem.

Homework 10.10
For $(A, +, 0)$ as in Homework 10.9, make a formal proof for

$$\{\forall x\,\forall y\,\forall z\,[(x + y) + z = x + (y + z)]\} \vdash \forall u\,[u + (u + u) = (u + u) + u]$$

using the axioms of Λ and the Generalisation Theorem.

Homework 10.11
Is the statement

$$\{\forall x\,\forall y\,\exists z\,[x + y = y + x], \forall x\,\forall y\,\forall z\,[(x + y) + z = x + (y + z)]\} \models \forall x\,\forall y\,\exists z\,[x + z = y]$$

true? If the statement is true then make a formal proof else provide a model satisfying the left but not the right side of $\models$.

Homework 10.12
Is the statement

$$\{\forall x\,\forall y\,\exists z\,[x + z = y], \forall x\,\forall y\,\forall z\,[(x + y) + z = x + (y + z)]\} \models \forall x\,\forall y\,\exists z\,[z + x = y]$$

true? If the statement is true then make a formal proof else provide a model satisfying the left but not the right side of $\models$.