MA 4207 - Mathematical Logic
Homework

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Homework 11.1
Assume that $\alpha, \beta, \gamma$ are well-formed formulas. Give a formal proof of the statement
$$\{\beta, \gamma\} \models \alpha \rightarrow \beta$$
which only uses the formulas from $\Lambda$ and the Modus Ponens.

Homework 11.2
Assume that $\{\alpha, \beta\}$ tautologically implies $\gamma$. The below derivation is incorrect. Say
what the fault is and replace it by a corrected one:

1. $\{\alpha, \beta\} \vdash \alpha \rightarrow \beta \rightarrow \gamma$ (Axiom Group 1)
2. $\{\alpha, \beta\} \vdash \beta$ (Copy)
3. $\{\alpha, \beta\} \vdash \beta \rightarrow \alpha \rightarrow \beta$ (Axiom Group 1)
4. $\{\alpha, \beta\} \vdash \alpha \rightarrow \beta$ (Modus Ponens)
5. $\{\alpha, \beta\} \vdash \gamma$ (Modus Ponens)

For the following exercises, $P, Q$ are predicates and $a, b, c$ are constants.

Homework 11.3
Make a formal proof for
$$\forall x [P(x) \rightarrow Q(c)], \forall x [\neg P(x) \rightarrow Q(c)] \vdash Q(c)$$

Homework 11.4
Make a formal proof for $\forall x [P(x)], \exists y [\neg P(y)] \vdash Q(z)$.

Homework 11.5
Make a formal proof for $\emptyset \vdash \forall x \forall y [P(x) \rightarrow Q(y)] \rightarrow P(a) \rightarrow Q(b)$.

Homework 11.6
Is the statement $\emptyset \vdash P(x) \rightarrow \forall y [P(y)]$ correct? Explain your answer.

Homework 11.7
Is the statement $\emptyset \vdash P(x) \rightarrow \forall y [P(x)]$ correct? Explain your answer.
Homework 11.8
Is the statement $\emptyset \vdash P(x) \rightarrow \exists y[P(y)]$ correct? Explain your answer.

Homework 11.9
Let $(G, \circ, f, e)$ be a structure and $\Gamma$ contain the following axioms:

- $\forall x, y, z \; [(x \circ y) \circ z = x \circ (y \circ z)];$
- $\forall x, y \; [x \circ y = y \circ x];$
- $\forall x \; [x \circ e = x];$
- $\forall x \; [x \circ f(x) = e];$
- $\forall x, y, z \; [x \circ y = x \circ z \rightarrow y = z];$

So $(G, \circ)$ is an Abelian group with neutral element $e$ and inversion $f$. Prove informally the following results:

- $\forall v, w \; [f(v) = f(w) \rightarrow v = w];$
- $\forall v, w \; [v \circ w = e \rightarrow f(v) = w];$
- $\forall v, w \; [f(v \circ w) = f(w) \circ f(v)].$

Homework 11.10
Consider all structures $(A, \circ)$ where $A$ has two elements and satisfies the axioms

$\forall x \; [x \circ x = x]$ and $\forall x \forall y \; [x \circ y = y \circ x].$

Show that all these structures are isomorphic.

Homework 11.11
Assume that $(\mathbb{N}, +, <, 0, 1, P)$ is a structure where $\mathbb{N}$ is the set of natural numbers and $+, <, 0, 1$ have the usual meaning on $\mathbb{N}$. Let the powers of 2 be the set $\{1, 2, 4, 8, 16, \ldots\}$ and make a formula $\alpha$ such that $(\mathbb{N}, +, <, 0, 1, P) \models \alpha$ iff $\forall x \; [Px \leftrightarrow x \text{ is a power of } 2].$

Note that such a formula only implicitly defines the powers of 2 and not explicitly; therefore this formula $\alpha$ does not say that the powers are definable from addition and order in $\mathbb{N}$.

Homework 11.12
Make a formula $\alpha$ which says that $f : A \rightarrow A$ is a one-to-one function but not an onto function. Provide a model $(A, f, =)$ which satisfies $\alpha$. Can $A$ be finite?