MA 4207 - Mathematical Logic
Course-Webpage `http://www.comp.nus.edu.sg/~fstephan/mathlogicug.html`
Homework

Frank Stephan. Departments of Mathematics and Computer Science,
10 Lower Kent Ridge Road, S17#07-04 and 13 Computing Drive, COM2#03-11,
National University of Singapore, Singapore 119076.
Email fstephan@comp.nus.edu.sg
Telephone office 65162759 and 65164246
Office hours Thursday 14.00-15.00h at Mathematics S17#07-04

Homework 12.1
Let α, β, γ be any well-formed formulas. Which of the below formulas are valid, independent of what formulas for α, β, γ are chosen? If yes, give a formal proof, if not, find a counter example by chosing the right values for α, β, γ.

1. $\forall x\,[\alpha \to \beta] \to \forall x\,[\neg\beta] \to \forall x\,[\neg\alpha]$;
2. $\forall x\,\forall y\,[\alpha \to \beta] \to \forall x\,\forall y\,[\beta \to \gamma] \to \forall x\,\forall y\,[\gamma \to \alpha]$;
3. $\forall x\,[\alpha \to \beta] \to \forall x\,[\alpha \to \neg\beta] \to \forall x\,[\neg\alpha]$.

Homework 12.2
Assume that x, y are different variables. Which of the below statements are valid for all choices of α:

- $\forall x\,[\alpha^x_y \to \alpha]$;
- $\forall x\,[\alpha \to \alpha^x_y]$.

Either provide a proof that the formula is valid or a counter example (one choice of α and the corresponding structure and default) where the formula is false.

Homework 12.3
Assume that the logical language contains the predicate symbols P and Q. Make formal proofs for the following facts. You can use the Deduction and the Generalisation Theorems and use axioms of the first group in order to deal with connectives other than $\neg$ and $\to$.

1. $\{P(y)\} \vdash \forall x\,[x = y \to P(x)]$;
2. $\{\forall x\,[x = y \to Q(x)], \forall z\,[Q(z) \to P(z)]\} \vdash P(y)$;
3. $\{\forall x\,[P(x)], \forall x\,[Q(x)]\} \vdash \forall x\,[P(x) \wedge Q(x)]$.

Homework 12.4
Prove the following statement, perhaps by first proving that $\{\forall y\,[\neg(y = f(x))]\}$ is inconsistent and then using that therefore $\neg\forall y\,[\neg(y = f(x))]$ can be proven from $\emptyset$:

$$\emptyset \vdash \forall x\,\exists y\,[y = f(x)].$$

Homework 12.5
If the following sentence is valid then prove it else provide a structure where it is false:

$$\forall x\, \exists y\, [f(f(x)) = y \wedge f(f(y)) = x].$$

Homework 12.6
If the following sentence is valid then prove it else provide a structure where it is false:

$$\exists y\, \forall x\, [y = f(x)] \rightarrow \exists y\, \forall x\, [y \neq f(x)].$$

Homework 12.7
Let the logical language contain an unary function f and constants a, b and equality. If the following sentence is valid then prove it else provide a structure where it is false:

$$f(a) \neq f(b) \rightarrow \forall x\, \exists y\, [f(x) \neq f(y)].$$

Homework 12.8
Let S, T be any sets of sentences and let the logical language contain $=$ and one unary function symbol f. Is the following true: If both S, T satisfy that they have models with six elements and these models are isomorphic then $S \cup T$ has also models with six elements and those are also all isomorphic. Prove your answer.

Homework 12.9
Let S, T be any sets of sentences. Is the following true: If all infinite structures are models of S and all infinite structures are models of T then all infinite structures are models of $S \cup T$?

Homework 12.10
Is the set

$$\{\forall x\, \forall y\, \exists z\, [x \circ z = y], \forall x\, \forall y\, \exists z'\, [z' \circ x = y],$$
$$\exists x\, \exists y\, \exists z\, [x \circ z = y \wedge z \circ x \neq y], \forall x\, \forall y\, \forall z\, [x \circ (y \circ z) = (x \circ y) \circ z]\}$$

a consistent set of formulas? In other words, is there a structure $(A, \circ)$ such that $\circ$ is associative and for each x, y one can find from each side elements z, z' such that $x \circ z = y$ and $z' \circ x = y$; however, it might be that for some x, y, z with $x \circ z = y$, one has to take a different z' for achieving $z' \circ x = y$.

Homework 12.11
Consider the finite structure with domain $\{0, 1, \ldots, p-1\}$ and multiplication and addition modulo p, besides the usage of $+, \cdot$ the logical language also permits $-$ and the constants $0, 1, \ldots, p-1$. Make in a programming language of your choice a computer program which evaluates to 0 (false) or 1 (true) depending on whether the statement

$$\forall x\, \exists y\, \exists z\, [x = y \cdot y + z \cdot z]$$

is true in the structure and use the program to determine for which of $p = 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12$ the formula is true, that is, for which of these modulo rings is every number the sum of two squares.