MA 4207 - Mathematical Logic
Course-Webpage `http://www.comp.nus.edu.sg/~fstephan/mathlogicug.html`
Homework

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Homework 14.1
Give an example for a set S of formulas in sentential logic such that

- for all $\alpha, \beta \in S$, the formulas $(\alpha \vee \beta), (\alpha \wedge \beta)$, $(\alpha \to \beta)$, $(\alpha \leftrightarrow \beta)$ and $\neg\neg\alpha$ are also in S;
- for all α, either $\alpha \in S$ or $\neg\alpha$ in S but not both.

Solution. Let v be such that $v(A) = 1$ for all atoms A. Now let

$$S = \{\alpha : \overline{v}(\alpha) = 1\}$$

and one can see that due to the definition of $\overline{v}$, it is always true that exactly one of $\alpha, \neg\alpha$ are in S. Furthermore, if α, β are in S then $\overline{v}$ makes both of them true and it follows that $\overline{v}$ also makes the formulas $(\alpha \vee \beta), (\alpha \wedge \beta)$, $(\alpha \to \beta)$, $(\alpha \leftrightarrow \beta)$ and $\neg\neg\alpha$ true, thus they are also in S.

Homework 14.2
How many Boolean functions of the form B^n_α can be built where α uses the atoms $A_1, \ldots, A_n$ and combines them either with $\wedge$ or with $\to$? Other connectives and logical constants are not allowed. List out the numbers of functions for $n = 1, 2, 3$.

Solution. The number of functions is 2^{2^n-1}. Note that $1 \to 1$ and $1 \wedge 1$ both evaluate to 1. Thus if $A_1, \ldots, A_n$ are all 1 then the output is 1. If at least one of them is 0, then $A_1 \wedge A_2 \wedge \ldots \wedge A_n$ has the value 0 and one can use this as a replacement for the constant 0; furthermore, $\neg\alpha$ is then realisable by the formulas $F(\alpha) = (\alpha \to (A_1 \wedge A_2 \wedge \ldots \wedge A_n))$. If at least one of the atoms is 0 then $F(\alpha)$ has the value $\neg\alpha$ else α has the value 1. It follows that one can translate a formula β using only $\wedge$ and $\neg$ – and every Boolean function in n variables can be represented by such a formula – and then one replaces in this formula all subformulas $\neg\alpha$ by $F(\alpha)$ until one gets a formula γ which only contains $\to$ and $\wedge$ and atoms. Now it holds that

$$B^n_\gamma = F^n_{\beta \vee (A_1 \wedge A_2 \wedge \ldots \wedge A_n)}$$

and therefore the given Boolean function is only changed to 1 in the case that all inputs are 1. For all other input-vectors, the original value is maintained. Thus one can choose $2^n - 1$ values freely and make the corresponding β and the number of such

$\{0,1\}$-valued functions is 2^{2^n-1}. For $n = 1,2,3,4$ these values are $2, 8, 128, 32768$. For $n = 1$, the two functions are the identity-function $B^1_{A_1}$ and the constant-1-function $B^1_{A_1 \to A_1}$.

Homework 14.3

Prove by induction that for every formula using only $\oplus$, $\leftrightarrow$ and $\neg$ as connectives, which is built from the atoms $A_1, A_2, \ldots, A_n$, either all possible assignments of these n values or half of them or none of them evaluates the formula to true.

Solution. What one is proving by induction is the following statement: Given a formula α using the above indicated connectives, one defines $\mathit{Depend}(\alpha)$ to be the set of all atoms A such that there are v, w assigning values to the atoms different only on A_k with $\overline{v}(A_k) \neq \overline{w}(A_k)$. One shows now by induction the following statement for formulas α of the given type:

$(*)$ If v, w differ exactly on A_k and $A_k \in \mathit{Depend}(\alpha)$ then $\overline{v}(\alpha) = \neg\overline{w}(\alpha)$.

To see $(*)$, first note that for constants, the sets $\mathit{Depends}(0)$ and $\mathit{Depends}(1)$ are empty and thus the statement is true; furthermore, if $\alpha = A_k$ then $\mathit{Depend}(A_k) = A_k$ and it is obvious that if v, w differ on A_k then $\overline{v}(A_k) = \neg\overline{w}(A_k)$. Now for an induction, consider α, β which are satisfying $(*)$:

1. Consider $\gamma = \neg\alpha$. One let $\mathit{Depend}(\gamma) = \mathit{Depend}(\alpha)$ and considers any v, w which differ only in one atom A_k: If $A_k \in \mathit{Depend}(\alpha)$ then $\overline{v}(\gamma) = \neg\overline{v}(\alpha) = \neg\neg\overline{w}(\alpha) = \neg\overline{w}(\gamma)$; If $A_k \notin \mathit{Depend}(\alpha)$ then $\overline{v}(\gamma) = \neg\overline{v}(\alpha) = \neg\overline{w}(\alpha) = \overline{w}(\gamma)$. So for v, w only differing in A_k, $\overline{v}(\gamma) = \neg\overline{w}(\gamma)$ iff $A_k \in \mathit{Depend}(\gamma)$.

2. Consider $\gamma = \alpha \oplus \alpha$. One let $\mathit{Depend}(\gamma)$ be the symmetric difference of $\mathit{Depend}(\alpha)$ and $\mathit{Depend}(\beta)$, that is, contain exactly those atoms which are in exactly one of the sets $\mathit{Depend}(\alpha)$ and $\mathit{Depend}(\beta)$. Consider any v, w which differ only in one atom A_k: If $A_k \in \mathit{Depend}(\gamma)$ then A_k is exactly in one of the sets $\mathit{Depend}(\alpha)$ and $\mathit{Depend}(\beta)$, without loss of generality say in the first. Now $\overline{v}(\alpha)$ and $\overline{w}(\alpha)$ differ while $\overline{v}(\beta)$ and $\overline{w}(\beta)$ are the same. It follows that one of $\overline{v}(\alpha \oplus \beta)$, $\overline{w}(\alpha \oplus \beta)$ is $0 \oplus c$ while the other one is $1 \oplus c$, where $c \in \{0,1\}$. Hence $\overline{v}(\gamma) = \neg\overline{w}(\gamma)$. If $A_k \notin \mathit{Depend}(\gamma)$ by $A_k \notin \mathit{Depend}(\alpha), A_k \notin \mathit{Depend}(\beta)$ then $\overline{v}(\alpha) = \overline{w}(\alpha)$, $\overline{v}(\beta) = \overline{w}(\beta)$ and $\overline{v}(\gamma) = \overline{v}(\alpha \oplus \beta) = \overline{w}(\alpha \oplus \beta) = \overline{w}(\gamma)$. If $A_k \notin \mathit{Depend}(\gamma)$ by $A_k \in \mathit{Depend}(\alpha), A_k \in \mathit{Depend}(\beta)$ then $\overline{v}(\alpha) = \neg\overline{w}(\alpha)$, $\overline{v}(\beta) = \neg\overline{w}(\beta)$ and $\overline{v}(\gamma) = \overline{v}(\alpha \oplus \beta) = \overline{w}(\neg\alpha \oplus \neg\beta) = \overline{w}(\alpha \oplus \beta) = \overline{w}(\gamma)$. So for v, w only differing in A_k, $\overline{v}(\gamma) = \neg\overline{w}(\gamma)$ iff $A_k \in \mathit{Depend}(\gamma)$.

3. If $\gamma = \alpha \leftrightarrow \beta$ then again $\mathit{Depend}(\gamma)$ is the symmetric difference of $\mathit{Depend}(\alpha)$ and $\mathit{Depend}(\beta)$. The proof in this case is the same as in the case of $\oplus$; alternatively, one could also replace $\alpha \leftrightarrow \beta$ by $\neg(\alpha \oplus \beta)$ and do the two prior inductive steps.

Thus the induction gives that for each formula α there are two cases: Either $\mathit{Depend}(\alpha) = \emptyset$ and then the truth-table of α assigns in all rows the same value or $\mathit{Depend}(\alpha)$

contains at least one atom A_k and if one puts this atom A_k into the last column of the truth-table and on can group the rows in pairs of rows where the truth-entries differ only for A_k and thus one of these rows carries the value 0 while the other one carries the value 1; hence half of the rows has a 0 and half has a 1. Here an example for $A_h \oplus \neg(A_k \oplus A_h)$:

A_h	A_k	$A_h \oplus \neg(A_k \oplus A_h)$
0	0	1
0	1	0
1	0	1
1	1	0

Homework 14.4
Make a formal proof that

$$\forall x\, \forall y\, [\alpha \rightarrow \beta] \rightarrow \forall y\, \forall x\, [\neg\beta \rightarrow \neg\alpha]$$

is a valid formula.

Solution. Recall that tautologies in sentential logic can be made to axioms in first-order logic by replacing the atoms by logical symbols; furthermore, any formula of the form $\forall x\, [\gamma] \rightarrow \gamma$ is an axiom. Thus one can make the following proof.

1. $\{\forall x\, \forall y\, [\alpha \rightarrow \beta]\} \vdash \forall x\, \forall y\, [\alpha \rightarrow \beta]$ (Copy);
2. $\{\forall x\, \forall y\, [\alpha \rightarrow \beta]\} \vdash \forall x\, \forall y\, [\alpha \rightarrow \beta] \rightarrow \forall y\, [\alpha \rightarrow \beta]$ (Axiom Group 2);
3. $\{\forall x\, \forall y\, [\alpha \rightarrow \beta]\} \vdash \forall y\, [\alpha \rightarrow \beta]$ (Modus Ponens);
4. $\{\forall x\, \forall y\, [\alpha \rightarrow \beta]\} \vdash \forall y\, [\alpha \rightarrow \beta] \rightarrow \alpha \rightarrow \beta$ (Axiom Group 2);
5. $\{\forall x\, \forall y\, [\alpha \rightarrow \beta]\} \vdash \alpha \rightarrow \beta$ (Modus Ponens);
6. $\{\forall x\, \forall y\, [\alpha \rightarrow \beta]\} \vdash (\alpha \rightarrow \beta) \rightarrow (\neg\beta \rightarrow \neg\alpha)$ (Axiom Group 1);
7. $\{\forall x\, \forall y\, [\alpha \rightarrow \beta]\} \vdash \neg\beta \rightarrow \neg\alpha$ (Modus Ponens);
8. $\{\forall x\, \forall y\, [\alpha \rightarrow \beta]\} \vdash \forall x\, [\neg\beta \rightarrow \neg\alpha]$ (Generalisation Theorem);
9. $\{\forall x\, \forall y\, [\alpha \rightarrow \beta]\} \vdash \forall y\, \forall x\, [\neg\beta \rightarrow \neg\alpha]$ (Generalisation Theorem);
10. $\emptyset \vdash \forall x\, \forall y\, [\alpha \rightarrow \beta] \rightarrow \forall y\, \forall x\, [\neg\beta \rightarrow \neg\alpha]$ (Deduction Theorem).

Homework 14.5
Is the statement

$$\{Px \rightarrow Py\} \vdash \forall z\, [Px \rightarrow Pz]$$

correct? If so, make a formal proof, if not, make a model with default values of the variables for which it is false.

Solution. This statement is not correct. Assume that a model is given with variable defaults, that the model has at least the values $0, 1$, that $P(x)$ is equivalent to $x = 0$ and that x, y have the default value 0. Then $Px \rightarrow Py$ is true and $Px \rightarrow P1$ is false; in particular $\forall z\,[Px \rightarrow Pz]$ is false.

Homework 14.6
Which of the following statements can be proven? If so, then give the formal prove, else explain why one cannot do it.

(a) $\{x = 0\} \vdash \forall x\,[x = 0]$;

(b) $\{\forall y\,[y = 0]\} \vdash \forall x\,[x = 0]$;

(c) $\{\forall y\,\forall x\,[x = y]\} \vdash \forall x\,[x = 0]$.

Solution. Statement (a) does not hold. The reason is that one can consider the model $(\mathbb{N}, 0)$ and then one sees, that a variable assignment can make $x = 0$ true while the conclusion that all values in the model equal to 0 is false. Note that the Generalisation Theorem can only be applied if the variable x does not occur free in the precondition.

Statement (b) can be proven along the same lines as one can prove the Principle of Alphabetical Variants, indeed, it follows from this principle directly. A formal proof, using only Axioms and the Generalisation Theorem, is the following:

1. $\{\forall y\,[y = 0]\} \vdash \forall y\,[y = 0]$ (Copy);
2. $\{\forall y\,[y = 0]\} \vdash \forall y\,[y = 0] \rightarrow x = 0$ (Axiom 2);
3. $\{\forall y\,[y = 0]\} \vdash x = 0$ (Modus Ponens);
4. $\{\forall y\,[y = 0]\} \vdash \forall x\,[x = 0]$ (Generalisation Theorem).

Here the Generalisation Theorem can be applied, as the variable x does not occur, actually does not occur at all, in the preconditions.

Statement (c) can also be proven and the proof is even easier, as one only needs axioms from Λ:

1. $\{\forall y\,\forall x\,[x = y]\} \vdash \forall y\,\forall x\,[x = y]$ (Copy);
2. $\{\forall y\,\forall x\,[x = y]\} \vdash \forall y\,\forall x\,[x = y] \rightarrow \forall x\,[x = 0]$ (Axiom 2);
3. $\{\forall y\,\forall x\,[x = y]\} \vdash \forall x\,[x = 0]$ (Modus Ponens).

Note that in formal proofs, the axioms from Λ and the copying from the preconditions and the usage of Modus Ponens are always allowed.

Homework 14.7
Choose a logical language and a theory T in this language such that

- T is finite axiomatisable;
- T is $\aleph_0$-categorical and $\aleph_1$-categorical;
- T has a finite model of m elements iff $m = 3^n$ for some n.

Furthermore, is T complete? Explain your answer.

Solution. The idea is to use the language of Abelian groups where an element three times added to itself gives 0. These structures are equivalent to vector spaces over $\mathbb{F}_3$ and it is known from linear algebra that each two such vector spaces are isomorphic iff they are vector spaces over the same field and their bases have the same cardinality. Note that scalar multiplication over $\mathbb{F}_3$ with 0 gives the constant 0 function and with 1 gives the identity function and with 2 gives the sum of an element with itself. Thus one can define scalar multiplication by cases and does not need to incorporate it into the logical language. So the only symbols added into the language are 0 (neutral element) and + (addition modulo 3 in a vector space). The axioms postulated are now the following ones:

1. $\forall x \, \forall y \, \forall z \, [(x + y) + z = x + (y + z)]$;
2. $\forall x \, \forall y \, [x + y = y + x]$;
3. $\forall x \, [x + 0 = x]$;
4. $\forall x \, [x + (x + x) = 0]$.

Now, if κ is an infinite cardinal then, by results of linear algebra, a vector space over $\mathbb{F}_3$ has a basis of cardinality κ iff the vector space itself has cardinality κ; thus every such vector space and, therefore, also every structure satisfying the above axioms is κ-categorical; in particular these structures are $\aleph_0$-categorical and $\aleph_1$-categorical. Furthermore, the finite vector spaces of dimension n have all 3^n elements and every finite vector space has a finite basis (dimension). It is not needed for this homework to reprove the facts known from basic lectures like linear algebra.

Homework 14.8 Consider the logical language containing one unary function f and the set

$$S = \{\forall x \, \forall y \, [x \neq y \rightarrow f(x) \neq f(y)], \, \forall x \, [x \neq f(x)], \, \forall x \, \exists y \, [f(y) = x]\}$$

and let $Th(S)$ be the set of all sentences which can be proven from S. Check whether the $Th(S)$ is 5-categorical, that is, whether all models of cardinality 5 of $Th(S)$ are isomormphic. Provide all models for $\kappa = 5$ and check to which κ this generalises.

Solution. The answer is that $Th(S)$ is not 5-categorical. There are two models, (a) the model of a 5-cycle and (b) the model of a 2-cycle plus a 3-cycle. So if one calls the elements $0, 1, 2, 3, 4$ and makes tables of f in the two models, the tables are the following:

Inputs	0	1	2	3	4
f in Model (a)	1	2	3	4	0
f in Model (b)	1	0	3	4	2

For other small κ, note that the theory is not 1-categorical, as it has no model of size 1. The theory is 2-categorical and 3-categorical, as these sizes permit only one cycle and that cycle is of length κ. The theory is not κ-categorical for any $\kappa \geq 4$, as one can make, for finite κ, (a) one κ-cycle and (b) one cycle of length 2 and one of length $\kappa - 2$ and for $\kappa \geq 6$ one can also further models. For infinite κ, it is not κ-categorical as one can make, for any $n \geq 2$, a model consisting of κ n-cycles.

Homework 14.9
Assume that the logical language contains one unary function f and equality $=$. Provide two sentences α and β such that the theories $Th(\{\alpha\})$ and $Th(\{\beta\})$ are κ-categorical for all $\kappa \geq 1$ and such that $Th(\{\alpha, \beta\})$ is complete and therefore either κ-categorical for exactly one finite κ or κ-categorical for exactly the infinite κ.

Solution.
The idea is to choose a finite κ such that the models of α and β coincide for exactly this κ and not for any other κ. Here one chooses $\kappa = 1$, as that is most easy to handle. Now the formulas are as follows: α is $\forall x\, \forall y\, [f(x) = f(y)]$; β is $\forall x\, [f(x) = x]$. So α requires that the function is constant and β requires that it is the identity. This can be combined if and only if the domain has exactly one element. Note that for $Th(\{\alpha\})$, two models of the same size are isomorphic, the isomorphism maps the elements in the two ranges to each other, as they are unique, and maps the other elements in a one-one way to each other. For two models of $Th(\{\beta\})$ of the same size, any bijection is an isomorphism, as the image of the identity-function is again the identity-function. $Th(\{\alpha, \beta\})$ has the unique model $(\{0\}, f, =)$ with $f(0) = 0$ (up to isomorphism) and therefore $Th(\{\alpha, \beta\})$ is complete. That all elements are equal in any model of $\{\alpha, \beta\}$ can be seen by the fact that given x, y, β implies that $x = f(x)$ and $y = f(y)$ and α implies that $f(x) = f(y)$ and thus $x = y$, so any two elements of the model are the same.

Please review also the old exams on the course homepage and other material available and read the course notes.