Members of thin $\Pi^0_1$ classes and their Turing degrees

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**Π^0_1** classes

**Definition:**
A Π^0_1 class is a set \( P \subseteq 2^\omega \) for which there is a (primitive) recursive tree \( T \) with \([T] = P\).

- As primitive recursive trees can be effectively enumerated, we have an effective enumeration of Π^0_1 classes.

**Examples:**

- Consider \( A = \{ e : \varphi_e(0) \downarrow = 0 \} \) and \( A = \{ e : \varphi_e(0) \downarrow = 1 \} \).
  - \( A \) and \( B \) are disjoint r.e. sets, and cannot be recursively separated.
  - The class of sets separating \( A \) and \( B \)
    \[ S(A, B) = \{ C : A \subseteq C \& B \cap C = \emptyset \} \]
    is called the separating class of \( A \) and \( B \).
  - \( S(A, B) \) is a Π^0_1 class and is perfect (hence uncountable).

- The class of complete consistent extension of Peano Arithmetic is a Π^0_1 class.

- Zariski topology over recursive rings, where for any r.e. ideal \( I \), the collection of prime ideals containing \( I \) forms a Π^0_1 class.

**Basis Theorems for Π^0_1 classes:** old friend
**Definition: Thin Classes**

A $\Pi^0_1$ class $P$ is thin if every subclass of $P$ is relatively clopen, i.e., if $Q$ is a subclass of $P$, then $Q = P \cap U$ for some clopen set $U \subseteq 2^\omega$.

We know that in all $\Pi^0_1$ classes, isolated paths are computable.

Conversely, if a thin $\Pi^0_1$ class $P$ contains a computable element $X$, then $\{X\}$ is a subclass of $P$, and hence by the thinness of $P$, $X$ is isolated.

**FACT:**

A thin $\Pi^0_1$ class $P$ has no computable members if and only if $P$ is perfect.

So every **countable thin $\Pi^0_1$ class** has a computable member.
Martin-Pour El theories

The notion of thinness comes from the work of Martin and Pour-El in 1970. Let $S$ be a consistent r.e. theory in the propositional language with

**Martin and Pour-El, 1970**

Let $S$ be a consistent r.e. theory.

1. $S$ has few r.e. extensions if each r.e. extension $T$ of $S$ is a principal extension, i.e., $T$ is generated by $S$ together with a single propositional formula.

2. $S$ is essentially undecidable if $S$ has no decidable complete consistent extensions.

**FACTS:**

For a consistent r.e. theory $S$,

- $S$ has few r.e. extensions if and only if the corresponding $\Pi^0_1$ class is thin.
- $S$ is essentially undecidable if and only if the corresponding $\Pi^0_1$ class has no computable members.
Theorem (CDJS, 1993):
If $X$ is in a thin $\Pi^0_1$ class $P$, then $X' \leq_T X \oplus \varnothing''$.

Proof: Let $P = [T]$ is a thin class, where $T$ is a recursive tree, and $A \in P$.
For a given $e$, we consider whether $e \in A'$ or not, i.e., whether $\Phi^A_e(e) \downarrow$ or not.

- If $\Phi^A_e(e) \downarrow$, we can recursive in $A$ to find an initial segment $\sigma$ of $A$ with $\{e\}^\sigma(e) \downarrow$.

- If NOT, what shall we do?
Consider $Q_e = \{ C : \Phi^C_e(e) \uparrow \}$, a $\Pi^0_1$ class

- $P \cap Q_e$ is a subclass of $P$, and as $P$ is thin, $P \cap Q_e = P \cap U_e$ for some clopen set $U_e$.
- As we are assuming that $A$ is in $P \cap Q_e$, $A$ is in $P \cap U_e$, and hence $A$ has an initial segment $\sigma$ with all infinite extensions in $U_e$.

Thus, if $B \in P$ extends $\sigma$, then $B \in P \cap U_e = P \cap Q_e$, and $\Phi^B_e(e) \uparrow$.

- Define a binary relation $R(e, \sigma)$ as

$$R(e, \sigma) \iff (\forall \tau \supseteq \sigma)[\tau \in T & \{e\}^\tau(e) \downarrow \rightarrow \tau \notin \text{Ext}(T)].$$

$R$ is a $\Pi^2$ relation and is recursive in $\phi''$.

We do as following:
Find the least number $n$ such the following is true for $\sigma = A \upharpoonright n$:

- (a) $\{e\}^\sigma(e) \downarrow \rightarrow e \in A'$
- (b) $R(e, \sigma) \rightarrow e \notin A'$

**Exact one of these** will appear.
Consider $Q_e = \{ C : \Phi^C_e(e) \uparrow \}$, a $\Pi^0_1$ class

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$$R(e, \sigma) \iff (\forall \tau \supseteq \sigma)[\tau \in T \& \{e\}^\tau(e) \downarrow \rightarrow \tau \notin \text{Ext}(T)].$$

$R$ is a $\Pi^2_2$ relation and is recursive in $\phi''$.

We do as following:

Find the least number $n$ such the following is true for $\sigma = A \upharpoonright n$:

(a) $\{e\}^\sigma(e) \downarrow \rightarrow e \in A'$
(b) $R(e, \sigma) \rightarrow e \notin A'$

Exact one of these will appear.

If $A$ computes $\phi''$, then $A$ cannot be a member of any thin $\Pi^0_1$ class.
In Spector’s construction of minimal degrees below $0''$, forcing notions are recursive perfect trees, $T_e, e \in \omega$, pruned according to the black-white rule.

That is, to see whether we can find a string $\sigma \in T_e$ such that there is no $e$-splitting above $\sigma$ in $T_e$, or not.

If we use only ‘half’ of each $T_e$, i.e., keep the even part, and exclude the odd part, the construction still works.
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A great observation.
Construct one thin-free degree below $0''$

**Definition:**
A Turing degree is thin-free, if no members in this degree is a member of thin \( \Pi^0_1 \) classes.

Note that all degrees above $0''$ are thin-free.

We will construct a set \( A \) of thin-free degree below $0''$, we shall ensure for any \( e \) such that if \( \Phi^A_e \) is total and Turing equivalent to \( A \), then one of the following is guaranteed:

1. \( \Phi^A_e \notin [P_e] \), or
2. \( [P_e] \) is not thin.
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The construction is modified from Spector’s construction of minimal degrees.

Suppose that $A$ is constructed on a given recursive perfect tree $T$.

- To meet (1), we try to find some string $\tau$ on $T$ such that $\Phi_e^\tau$ is not extendible on $P_e$,
  - if such a $\tau$ exists, we force $A$ to extend $\tau$, which guarantees that $\Phi_e^A \notin [P_e]$, if $\Phi_e^A$ is total.
If NOT, we will then try to

- find a $\Pi^0_1$ subclass of $[P_e]$ which is not the intersection of $[P_e]$ with any clopen set $U$.

We will construct a recursive subtree $S_e$ of $P_e$, such that $\Phi^A_e$ lies on $S_e$, and for any length $n$, there exists some $B \in [S_e]$ and $C \in [P_e] \setminus [S_e]$ such that

$$B \upharpoonright n = C \upharpoonright n = \Phi^A_e \upharpoonright n.$$ 

This implies that $\Phi^A_e \in [P_e]$, and $[S_e]$ witnesses that $P_e$ is not thin.
Action under this case:

- **Target**: Force $A$ on a total recursive subtree $T_e$ of $T$, such that for any $\alpha \in T_e$, $\Phi_{T_e(\alpha_0)}^e$ and $\Phi_{T_e(\alpha_1)}^e$ are incompatible in $P_e$ and there is a path on $P_e$ extending $\Phi_{T_e(\alpha)}^e$, of course.

  We are assuming that (1) fails, so both $\Phi_{T_e(\alpha_0)}^e$, $\Phi_{T_e(\alpha_1)}^e$ are extendible on $P_e$ and thus there is at least one infinite path in $P_e$ extending it.

Consider the $e$-splitting subtree of $T$, $SP(T, e)$, if exists, and take the even part.

- **White Side**: $SP(T, e)$ exists.

  In this case, $\Phi_e^A$ is total, then $\Phi_e^A$ is on $[P_e]$, and $E(SP(T, e))$, the even subtree of $SP(T, e)$, is a total recursive subtree of $T$, and $\Phi_e^{E(SP(T, e))}$ is a total recursive subtree of $P_e$, witnessing that $[P_e]$ is not thin.

- **Black Side**: $SP(T, e)$ does not exist.

  In this case, there is a string $T(\alpha)$ such that above $T(\alpha)$, no string $e$-splits, and hence, if $A$ is on the full subtree of $T$ above $\alpha$, $Full(T, \alpha)$, then $\Phi_e^A$ is recursive, making $A$ and $\Phi_e^A$ not Turing equivalent, if we can make $A$ nonrecursive. **We Can**, as recursive sets are all in thin $\Pi_1$ classes.
Oracle Construction:

We can now run a forcing argument to construct $A$ with wanted property.
   - $0''$ is used as oracle to make decision at every stage.

Yuan Bowen improved this in his thesis:

**Theorem:**
There exists a hyperimmune-free minimal degree below $0''$ which is also thin-free.

Note that such degrees are not below $0'$. 
Working below $0'$

- CDJS proved that $0'$ contains a $\Pi_1$ set $A$ which is in a thin $\Pi_1$ class $P$.

- CDJS proved the density of degrees containing sets (not necessarily r.e.) in thin $\Pi_1$ classes in r.e. degrees.

  DWY strengthened this in 2018, showing that sets above can be r.e.

- Yuan Bowen proved in his thesis that all 1-generic degrees below $0'$ contain members of thin $\Pi_1$ classes.
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- There are degrees below $0'$ thin-free and then can be r.e., or minimal, by CDJS.

  The construction of a minimal thin-free degree was given by CDJS, modified from Sacks forcing, where partial recursive trees are used.
An r.e. thin-free degree

Construct an r.e. \( A \) satisfying the following requirements:

\( \mathcal{R}_e \): if \( \Phi_e(A) \) and \( \Psi_e(\Phi_e(A)) \) are both total, then either

- \( A \neq \Psi_e(\Phi_e(A)) \); or
- \( \Phi_e(A) \) is not in \( [P_e] \); or
- \( [P_e] \) is not thin.

In this construction, we cannot use the e-splitting tree as a help to construct a subclass witness that \( [P_e] \) is not thin.

We thus need to construct such a subclass, actually, a subtree, by infinitely many substrategies, each of which tries to find an infinite path in \( [P_e] \), and

- any substrategy fails to secure an infinite path, an enumeration of a certain number into \( A \), showing that either \( A \neq \Psi_e(\Phi_e(A)) \) (diagonalization succeeds) or \( \Phi_e(A) \) is not in \( [P_e] \), a global win for \( \mathcal{R}_e \).

DWY proved in 2018 that such r.e. degrees are dense in the r.e. degrees.
Other topics

In his thesis, Yuan proved that any nonrecursive set below a 2-generic set is thin-free. In particular, 2-generic degrees are thin-free.

CDJS also consider minimal $\Pi^0_1$ classes and Cantor-Bendixson rank of sets, a topic originated from Cenzer, et al.'s work in 1986.

Our continuing work on this topic is in the direction of Ershov hierarchy, also 1-generic degrees not below $0'$, $pb$-generic degrees, minimal degrees with full approximations.

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4. Downey, Wu and Yang, Degrees containing members of thin Pi01 classes are dense and co-dense, Journal of Mathematical Logic, 18 (2018), DOI: 10.1142/S0219061318500010.

Thanks!
Thanks!

Take care and keep safe!