# Logic Day 2021

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# Remembering C. E. Michael Yates

On 21 Dec 2020, Mike Yates died aged 81 of cancer. He has been studying (PhD 1963 under Robin Gandy and, informally, John Shepherdson and Martin Davis) and working until 1989 at the University of Manchester. Afterwards he dedicated himself to the teaching of mathematics with assistance of computers and taught at Leeds Metropolitan University, Liverpool John Moores University and the University at Bangor (in Wales). Prior to starting his scientific career in Manchester, Mike Yates spent several years in the USA at Cornell and at the Institute of Advance Studies in Princeton.

Most remembered for groundbreaking work in Recursion Theory in the 1960ies and 1970ies. It includes the study of minimal pairs and the study of coretraceable recursively enumerable sets.

# Early work on Retracing Functions

His early work (1962) dealt with retraceable sets $A = \{a_1, a_2, \ldots\}$, these are sets where there is a partial recursive function $\mathrm{f}$ such that for all $n$, $f(a_{n+1}) = a_n$. Mike Yates published in 1962 three results:

(a) He gave a characterisation of the coretraceable r.e. sets.

(b) He constructed a retracing function $f$ for some set such that this function is not the extension of a finite-one retracing function.

(c) The hyperhypersimple sets are exactly those r.e. sets whose complement is infinite but does not have any infinite retraceable subset. Here a set $A$ is hyperhypersimple iff it is r.e., has an infinite complement and there is no uniformly r.e. array of disjoint r.e. sets $B_n$ such that each $B_n$ intersects the complement of $A$.

# Minimal Pairs

In the Turing degrees there are minimal degrees; however, for the recursively enumerable Turing degrees, one can construct below every nonrecursive r.e. degree another nonrecursive r.e. degree and there are well-known splitting theorems which split every recursively enumerable set into two disjoint subsets of strictly lower degree.

Mike Yates (and independently Alistair Lachlan) discovered in 1966 minimal pairs: There are nonzero r.e. degrees $\mathbf{a}$ and $\mathbf{b}$ such that the recursive degree $\mathbf{0}$ is the only r.e. degree below both. This result confirms a conjecture of Gerald Sacks.

Furthermore, Mike Yates showed that there is a Turing incomplete and nonzero r.e. degree $\mathbf{a}$ such that no nonzero r.e. degree $\mathbf{b}$ forms with $\mathbf{a}$ a minimal pair, that is, all nonzero r.e. degrees $\mathbf{b}$ satisfy that there is a nonzero common lower bound $\mathbf{c}$ below $\mathbf{a}$ and $\mathbf{b}$.

# Minimal degrees and r.e. degrees

Mike Yates showed in 1970 that below every nonzero r.e. degree $\mathbf{a}$, there is a (non-r.e.) minimal degree $\mathbf{b}$, that is, strictly below $\mathbf{b}$ there is exactly one degree which is $\mathbf{0}$.

An application of this result is that for all nonzero r.e. degree $\mathbf{a}$ the structures of the r.e. degrees below $\mathbf{a}$ and all Turing degrees below $\mathbf{a}$ are not elementary equivalent.

Besides his scientific work, Jeff Paris also mentioned that he was a distinguished rock climber who was, as a student, a member of a team which did several first climbs (of difficult rock formations) in the Lake district, an area in North-West England.

# Selected Results of Frank Stephan

Frank Stephan worked in Recursion Theory, Inductive Inference, Algorithmic Randomness, Automata Theory and Games on Finite Graphs. He did his PhD thesis about a generalisation of topology, but published in this field only one paper, later with Carl Mummert who had given a list of open questions as a starting point of their collaboration.

Cohesive Sets. A set $A$ is cohesive iff it is infinite and every r.e. set $W$ satisfies that either $A \cap W$ or $A - W$ is finite. Jockusch and Stephan constructed a cohesive set which is not high, that is, whose degree $\mathbf{a}$ does not satisfy $\mathbf{a}' \geq \mathbf{k}'$ where $\mathbf{k}$ is the degree of the halting problem. In fact such a degree can be low$_2$, that is, $\mathbf{a}'' \leq \mathbf{k}'$. However, as already Cooper showed in prior work, $\mathbf{a} \leq \mathbf{k}$ implies $\mathbf{a}' \geq \mathbf{k}'$ for cohesive sets.

# Structure inside truth-table degrees

Truth-table reduction: Turing reduction total for all oracles. Bounded Truth-Table reduction: tt-reduction where for each $x$ only constantly many queries are made to the oracle. Positive reduction: tt-reduction $F$ satisfying $F(A) \subseteq F(B)$ whenever $A \subseteq B$.

For this paper, all degrees are nonrecursive.

Dëgtev's Question: How many btt-degrees are in a tt-degree? Preliminary result: At least two.

Answer: Infinitely many btt-degrees. There are infinite chains and infinite antichains of btt-degrees in every tt-degree.

Corollary: Truth-table degrees contain infinite antichains of many-one degrees, answering a question of Jockusch.

# Positive degrees inside tt-degrees

Jockusch 1969: There are at least three positive degrees in a truth-table degree.

Results: (1) There is an r.e. tt-degree consisting of three positive degrees; one positive degree consists entirely of semirecursive and r.e. sets; one entirely of semirecursive and co-r.e. sets; one of nonsemirecursive non-r.e. sets.

(2) The number of positive degrees in a tt-degree is either odd or infinite.

(3) Given $n \geq 1$, let $m$ be the number of partial, transitive and irreflexive orderings on $\{0, 1, \ldots, n\}$. Then there is a nonrecursive tt-degree consisting of exactly $m$ positive degrees. In particular, there are tt-degrees consisting of $3$, $19$, $219$, $4231$, $130023$ and $6129859$ positive degrees.

Open Question: Are other odd numbers above $5$ possible numbers of positive degrees inside tt-degrees?

# Algorithmic Randomness

Question of Ambos-Spies and Kučera 2000: Characterise the Turing degrees such that all degrees above it contain a Martin-Löf random set.

Prior Result of Kučera 1985 and Gács 1986: All Turing degrees above the halting problem contain a Martin-Löf random set.

Prior Result of Kučera 1989: The Turing degrees of Martin-Löf random sets are not closed upwards. Kučera constructed a degree $\mathbf{a} < \mathbf{k}$ such that $\mathbf{a}$ contains a Martin-Löf random and all $\mathbf{b}$ with $\mathbf{a} \leq \mathbf{b}$ and $\mathbf{k} \not\leq \mathbf{b}$ contain either a Martin-Löf random or a PA-complete but not both.

Findings: A Martin-Löf random set is either PA-incomplete or above the halting problem. As every degree $\mathbf{a} \not\geq \mathbf{k}$ is bounded by a PA-complete degree $\mathbf{b} \not\geq \mathbf{k}$, there are Turing degrees above $\mathbf{a}$ not containing a Martin-Löf random set.