Generalisations of a result by Gul'ko on spaces of continuous functions

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Spaces of continuous functions

Let X be a Tychonov space.

Define $C(X) = \{f : X \to \mathbb{R} : f \text{ is continuous}\} \subseteq \mathbb{R}^X$.

Endow $\mathbb{R}^X$ with the product topology.

We denote $C(X)$ as a subspace of $\mathbb{R}^X$ by $C_p(X)$.

$C_p(X) \subseteq \mathbb{R}^X$ is a

- topological space.
- topological vector space (linear space).
- topological ring.

Functional Equivalences

Let X and Y be spaces

- If $C_p(X)$ and $C_p(Y)$ are homeomorphic, then X and Y are defined to be *t-equivalent*. Notation $X \overset{t}{\sim} Y$
- If $C_p(X)$ and $C_p(Y)$ are linearly homeomorphic, then X and Y are defined to be *l-equivalent*. Notation $X \overset{l}{\sim} Y$

Fact: We have $X \approx Y \Rightarrow X \overset{l}{\sim} Y \Rightarrow X \overset{t}{\sim} Y$

Theorem: [Nagata, 1949]

If topological rings $C_p(X)$ and $C_p(Y)$ are topologically isomorphic, then X and Y are homeomorphic.

Functional Equivalences

Theorem: [Dobrowolski, Gul'ko & Mogilski, 1990]

All metrizable countable non-discrete spaces are t-equivalent.

Theorem: [Arkhangelsk'ii, 1982]

If X and Y are l-equivalent, then X is compact iff Y is compact.

Corollary: $\mathbb{Q}$ and $\omega + 1$ are t-equivalent but not l-equivalent.

Corollary: $X \overset{t}{\sim} Y \not\Rightarrow X \overset{l}{\sim} Y$.

Examples:

- $\omega + 1$ and $\omega^2 + 1$ are l-equivalent and
- ω^2 and ω^ω are l-equivalent.

Corollary: $X \overset{l}{\sim} Y \not\Rightarrow X \approx Y$.

Linear invariant properties

Define a topological property $\mathcal{P}$ to be l-**invariant** if for l-equivalent spaces X and Y we have X has property $\mathcal{P}$ iff Y has property $\mathcal{P}$.

Some examples

1. For Tychonov spaces:
   - Compact, pseudocompact (Arkhangel'skii, 1982)
   - Lindelöf (Velichko, 1998)
   - Dimension (Gul'ko, 1993)
2. For metric spaces:
   - Locally compact, scattered (Baars & de Groot, 1992)
   - Čech complete (Baars, de Groot & Pelant 1993)
3. Open problem for countably compactness

Classification of function spaces

General question:

What are the common properties that two spaces need to have to be l-equivalent (or t-equivalent)?

- They need to have the same cardinality.
- If $|X| = |Y| = n$, then $C_p(X) \approx C_p(Y) \approx \mathbb{R}^n$.
- For countably infinite spaces the situation is not clear, but partial results are found
- For X countable, since $C_p(X) \subseteq \mathbb{R}^X$ we have $C_p(X)$ is metrizable even X is not.

Countably infinite spaces

Theorem: [Dobrowolski, Marciszewski & Mogilski, 1991]
All countable non-discrete spaces X for which $C_p(X)$ is $\mathcal{F}_{\sigma\delta}$ are t-equivalent.

In particular this holds for all metrizable countable non-discrete spaces. For l-equivalent spaces the situation is quite different.

For linear equivalence, a complete classification has been found for

- Countable compact metrizable spaces.
- Countable locally compact metrizable spaces.
- Countable metrizable spaces of scattered height $\leq \omega$.

Countably infinite spaces with only one non-isolated point

Let $\mathcal{A} = \{X : |X| = \omega \wedge \text{ only one } x \in X \text{ is not isolated}\}$.

Let $\mathcal{F}$ be a free filter on ω and define $\omega_{\mathcal{F}} = \omega \cup \{\infty\}$, where

- Each element of ω is isolated
- $\{F \cup \{\infty\} : F \in \mathcal{F}\}$ is a neighborhood base for ∞

Then $X \in \mathcal{A}$ iff there is $F \in \mathcal{F}$ such that $X \approx \omega_{\mathcal{F}}$.

We assume $\mathcal{A} = \{\omega_{\mathcal{F}} : F \in \mathcal{F}\}$

Let $\mathcal{B} = \{\omega_{\mathcal{F}} \in \mathcal{A} : \mathcal{F} \text{ is a free ultrafilter on } \omega\} \subseteq \mathcal{A}$

Then $X \in \mathcal{B}$ iff there is $u \in \omega^* = \beta\omega \setminus \omega$ such that

$X \approx \omega_u = \omega \cup \{u\}$.

We assume $\mathcal{B} = \{\omega_u : u \in \omega^*\}$

Filters and ultrafilters on ω

$\mathcal{A} = \{\omega_{\mathcal{F}} : F \in \mathcal{F}\}$, $\mathcal{B} = \{\omega_u : u \in \omega^*\}$ and $\mathcal{B} \subseteq \mathcal{A}$

Question: Let $X, Y \in \mathcal{A}$ be l-equivalent spaces. Are X and Y homeomorphic?

Theorem: [Gul'ko, 1990]

If $X, Y \in \mathcal{B}$ are l-equivalent spaces, then $X \approx Y$.

Other claims by Gul'ko

1. If $X, Y \in \mathcal{A}$ are l-equivalent spaces, then $X \in \mathcal{B}$ iff $Y \in \mathcal{B}$.
2. If $X \in \mathcal{B}$ and $n, m \in \omega$, then $C_p(X)^n$ and $C_p(X)^m$ are linearly homeomorphic iff $n = m$.

Unclear hint of a proof for (1) and the proof of (2) was not correct.

Some observations

Observation 1:

Let $X_1, X_2 \in \mathcal{A}$. Then $X_1 \oplus X_2$ has two non-isolated points.

Define Y to be the quotient space of $X_1 \oplus X_2$ by identifying the non-isolated points of X_1 and X_2.

Then $Y \in \mathcal{A}$ and $X_1 \oplus X_2 \overset{l}{\sim} Y$.

Observation 2:

Let $X \in \mathcal{A} \setminus \mathcal{B}$ and let ∞ be the only non-isolated point.

Then $X = Y_1 \cup Y_2$, where $Y_1 \cap Y_2 = \{\infty\}$ and ∞ is non-isolated in Y_1 and Y_2.

We have $Y_1, Y_2 \in \mathcal{A}$ and $X \overset{l}{\sim} Y_1 \oplus Y_2$.

Generalisations of Gul'ko's result

Theorem 1:

If $X = \bigoplus_{i=1}^{n} X_i$ and $Y = \bigoplus_{i=1}^{m} Y_i$ are l-equivalent spaces with each $X_i \in \mathcal{B}$ and each $Y_i \in \mathcal{A}$, then

- $m \leq n$ and if $m = n$, then each $Y_i \in \mathcal{B}$.

Theorem 2:

If $X = \bigoplus_{i=1}^{n} X_i$ and $Y = \bigoplus_{i=1}^{n} Y_i$ are l-equivalent spaces with each $X_i, Y_i \in \mathcal{B}$, then there is a permutation

$\pi : \{1, \cdots, n\} \to \{1, \cdots, n\}$ such that each $X_i \approx Y_{\pi(i)}$

In particular X and Y are homeomorphic.

Corollary: Gul'ko's claims follow from Theorem 1.

The support function

Let $L(X) = \{F : C_p(X) \to \mathbb{R} : F \text{ is a linear functional}\}$.

For $x \in X$ define $\xi_x : C_p(X) \to \mathbb{R}$ by $\xi_x(f) = f(x)$.

Then $\{\xi_x : x \in X\}$ is a Hamel basis for $L(X)$.

Let $\phi : C_p(X) \to C_p(Y)$ be a continuous linear function.

For $y \in Y$ define $\psi_y \in L(X)$ by $\psi_y(f) = \phi(f)(y)$.

There are $x_1, \cdots, x_n \in X$ and $\lambda_1, \cdots, \lambda_n \in \mathbb{R} \setminus \{0\}$ such that

$$\psi_y = \sum_{i=1}^{n} \lambda_i \xi_{x_i}.$$

Then for every $f \in C_p(X)$ we have

$$\phi(f)(y) = \sum_{i=1}^{n} \lambda_i f(x_i).$$

We define the *support of* y by $\mathrm{supp}_\phi(y) = \{x_1, \cdots, x_n\}$.

Properties of the support function

1. If $f = g$ on $\mathrm{supp}_\phi(y)$, then $\phi(f)(y) = \phi(g)(y)$.
2. Let $\phi : C_p(X) \to C_p(Y)$ be a continuous linear homeorphism.

   For $y \in Y$ we have $y \in \mathrm{supp}_{\phi^{-1}}(\mathrm{supp}_\phi(y))$.

   Hence there is $x \in \mathrm{supp}_\phi(y)$ such that $y \in \mathrm{supp}_{\phi^{-1}}(x)$.

Proof of (2):

Suppose $y \notin \mathrm{supp}_{\phi^{-1}}(\mathrm{supp}_\phi(y))$. Find $g \in C_p(Y)$ such that

$$g(y) = 1 \text{ and } g(\mathrm{supp}_{\phi^{-1}}(\mathrm{supp}_\phi(y))) = 0.$$

Let $f \in C_p(X)$ be such that $\phi(f) = g$.

Then $g = 0$ on $\mathrm{supp}_{\phi^{-1}}(\mathrm{supp}_\phi(y))$.

Hence by (1), $\phi^{-1}(g) = f = 0$ on $\mathrm{supp}_\phi(y)$.

Again by (1), $\phi(f)(y) = g(y) = 0$. Contradiction.

The θ function

Let $\phi : C_p(X) \to C_p(Y)$ be a linear homeorphism.

Define $\theta_\phi(y) = \{x \in \mathrm{supp}_\phi(y) : y \in \mathrm{supp}_{\phi^{-1}}(x)\}$.

Then $\theta_\phi(y) \neq \emptyset$.

Note that $x \in \theta_\phi(y)$ iff $y \in \theta_{\phi^{-1}}(x)$.

The following lemma is generalized version of a result by Gul'ko.

Main Lemma: *Let X and Y be spaces and let $\phi : C_p(X) \to C_p(Y)$ be a linear homeomorphism. Let B be a countable discrete clopen subset of X and let A be a countable subset of Y such that for every $y \in A$, $\theta_\phi(y) \cap B \neq \emptyset$. Then A is closed and discrete in Y.*

A property of the θ function

Lemma:

Let X and Y be spaces and let $\phi : C_p(X) \to C_p(Y)$ be a linear homeomorphism. Then for every $y \in Y$ we have

$$\sum\{\lambda_z^y \mu_y^z : z \in \theta_\phi(y)\} = 1.$$

Proof:

Let $g \in C_p(Y)$ be such that $g(y) = 1$ and $g(\mathrm{supp}_{\phi^{-1}}(\mathrm{supp}_\phi(y)) \setminus \{y\}) = 0$. Then

$$\begin{aligned} 1 = g(y) &= \sum\{\lambda_z^y \phi^{-1}(g)(z) : z \in \mathrm{supp}_\phi(y)\} \\ &= \sum\{\lambda_z^y \mu_w^z g(w) : z \in \mathrm{supp}_\phi(y) \wedge w \in \mathrm{supp}_{\phi^{-1}}(z)\} \\ &= \sum\{\lambda_z^y \mu_y^z : z \in \theta_\phi(y)\}. \end{aligned}$$

Theorem 1: Sketch of a proof for $n = 1$

Let $X = \omega_u \in \mathcal{B}$ and $\mathcal{U}$ the corresponding ultrafilter.

Let $Y, Z \in \mathcal{A}$ be with one non-isolated point.

Assume $\phi : C_p(X) \to C_p(Y \oplus Z)$ is a linear homeomorphism.

For $x \in X$, let $T(x) = \theta_\phi(\theta_{\phi^{-1}}(x))$.

Claim: $U = \{x \in \omega : |T(x)| = 1\} \in \mathcal{U}$.

Proof: For $y \in \theta_{\phi^{-1}}(x)$ we have $x \in \theta_\phi(y)$, hence $x \in T(x)$.

Suppose $U \notin \mathcal{U}$. Then $V = \{x \in \omega : |T(x)| > 1\} \in \mathcal{U}$.

For $x \in V$, pick $\sigma(x), \xi(x) \in T(x)$ with $\sigma(x) \neq \xi(x)$.

Let $C = \{\sigma(x) : x \in V\}$ and $D = \{\xi(x) : x \in V\}$.

Pick $\tau(x) \in Y \oplus Z$ such that $\tau(x) \in \theta_{\phi^{-1}}(x)$ and $\sigma(x) \in \theta_\phi(\tau(x))$

Pick $\kappa(x) \in Y \oplus Z$ such that $\kappa(x) \in \theta_{\phi^{-1}}(x)$ and $\xi(x) \in \theta_\phi(\kappa(x))$

Theorem 1: Sketch of a proof for $n = 1$

Let $A = \{\tau(x) : x \in V\}$

By the main lemma, A is not closed and discrete.

Since $C = \{\sigma(x) : \tau(x) \in A\}$, by the main lemma $C \in \mathcal{U}$.

Define $\pi : \omega \to \omega$ by $\pi(\sigma(x)) = \xi(x)$ and $\pi(x) \neq x$ elsewhere.

Then π has no fixed points. By a result of Katetov,

$\omega = Z_1 \cup Z_2 \cup Z_2$ with $\pi(Z_i) \cap Z_i = \emptyset$.

Assume $Z_1 \in \mathcal{U}$. Then $E = C \cap Z_1 \in \mathcal{U}$ and hence $\pi(E) \notin \mathcal{U}$.

Let $B = \{\kappa(x) : \sigma(x) \in E\}$ and $D = \{\xi(x) : \kappa(x) \in B\}$.

By the main lemma B is not closed and discrete and $D \in \mathcal{U}$.

For $\xi(x) \in D$, we have $\sigma(x) \in E$, hence $D \subseteq \pi(E)$. Contradiction.

This proves the claim.

Theorem 1: Sketch of a proof for $n = 1$

We conclude that $U = \{x \in \omega : |T(x)| = 1\} \in \mathcal{U}$

Let $V = \{x \in \omega : \theta_{\phi^{-1}}(x) \cap Y \neq \emptyset\}$ and

$W = \{x \in \omega : \theta_{\phi^{-1}}(x) \cap Z \neq \emptyset\}$.

By the main lemma, $V \in \mathcal{U}$ and $W \in \mathcal{U}$.

Hence $U \cap V \cap W \in \mathcal{U}$.

Pick $x \in U \cap V \cap W$ and let $Q = \theta_{\phi^{-1}}(x)$. Then $|Q| \geq 2$.

For every $y \in Q$ we have $\theta_\phi(y) = \{x\}$.

For every $y \in Q$ we have $\sum\{\lambda^y_z \mu^z_y : z \in \theta_\phi(y)\} = \lambda^y_x \mu^x_y = 1$.

We also have $1 = \sum\{\lambda^y_x \mu^x_y : y \in Q\} \geq 2$. Contradiction.

Theorem 2: Sketch of a proof

For $i \leq n$, let $\omega_{u_i}, \omega_{v_i} \in \mathcal{B}$ with $\mathcal{U}_i, \mathcal{V}_i$ the corresponding ultrafilters.

Assume $\phi : C_p(\oplus_{i=1}^n \omega_{u_i}) \to C_p(\oplus_{i=1}^n \omega_{v_i})$ is a linear homeomorphism.

Define $U_1 \subseteq \omega_{u_1}$ by $U_1 = \{x \in \omega : \theta_{\phi^{-1}}(x) \cap \omega_{v_1} \neq \emptyset\}$.

Suppose $U_1 \in \mathcal{U}_1$. There is $f : \omega_{u_1} \to \omega_{v_1}$ such that $f(x) \in \theta_{\phi^{-1}}(x) \cap \omega_{v_1}$ for $x \in U_1$ and $f(u_1) = v_1$.

By the main lemma, f is continuous and $f(U_1) \in \mathcal{V}_1$.

Define $V_1 = \{y \in f(U_1) : \theta_\phi(y) \cap \omega_{u_1} \neq \emptyset\}$. Then $V_1 \in \mathcal{V}_1$.

As above, there is a continuous $g : \omega_{v_1} \to \omega_{u_1}$ with $g(v_1) = u_1$.

A result on the Rudin-Keisler order on $\beta\omega$ gives $\omega_{u_1} \approx \omega_{v_1}$.

Theorem 2: Sketch of a proof

By the main lemma for each $i \leq n$, there is $j \leq n$ such that

$U_i^j = \{x \in \omega \subseteq \omega_{u_i} : \theta_{\phi^{-1}}(x) \cap \omega_{v_j} \neq \emptyset\} \in \mathcal{U}_i$ and hence $\omega_{u_i} \approx \omega_{v_j}$.

Partition $\{1, \cdots, n\}$ by $\{A_1, \cdots, A_N\}$ and $\{B_1, \cdots, B_N\}$ such that for each $k \leq N$, $i \in A_k$ and $j \in B_k$ we have $\omega_{u_i} \approx \omega_{v_j}$.

For the required permutation we need to show that $|A_k| = |B_k|$.

To illustrate this assume $A_1 = \{1, 2\}$ and $B_1 = \{1, 2, 3\}$.

Let $U_1 = U_1^1 \cap U_1^2 \cap U_1^3 \in \mathcal{U}_1$ and $U_2 = U_2^1 \cap U_2^2 \cap U_2^3 \in \mathcal{U}_2$.

Then $V_1 = \{x \in U_1 : |T(x) \cap U_1| = 1 \wedge |T(x) \cap U_2| = 1\} \in \mathcal{U}_1$.

Theorem 2: Sketch of a proof

Pick $x_1 \in V_1$.

Let $x_2 \in T(x) \cap U_2$ be such that $T\{x_1, x_2\} = \{x_1, x_2\}$.

Let $Q = \theta_{\phi^{-1}}(\{x_1, x_2\})$. Then $|Q| \geq 3$.

For every $y \in Q$ we have $\theta_\phi(y) = \{x_1, x_2\}$

and $\sum\{\lambda_z^y \mu_y^z : z \in \theta_\phi(y)\} = 1$.

Also $\sum\{\lambda_{x_1}^y \mu_y^{x_1} : y \in \theta_{\phi^{-1}}(x_1)\} = 1$

and $\sum\{\lambda_{x_2}^y \mu_y^{x_2} : y \in \theta_{\phi^{-1}}(x_2)\} = 1$.

Then $2 = \sum\{\lambda_x^y \mu_y^x : x \in \{x_1, x_2\} \wedge y \in Q\} \geq 3$. Contradiction.

An example

Gul'ko: If $u, v \in \omega^*$ and $\omega_u \overset{l}{\sim} \omega_v$, then $\omega_u \approx \omega_v$.

Question: Let α be a limit ordinal. Suppose $u, v \in \alpha^* = \beta\alpha \setminus \alpha$ and $\alpha_u \overset{l}{\sim} \alpha_v$. Is it always true that $\alpha_u \approx \alpha_v$?

Answer: No

Let $X = \omega^2$ and for $n < \omega$, $X_n = \omega + 1$. Then $X \approx \bigoplus_{n<\omega} X_n$.

For $n < \omega$, let z_n be the non-isolated point in X_n.

Let $D = \{z_n : n < \omega\}$. Then $\mathrm{cl}_{\beta X} D$ of D in βX is $\beta D \approx \beta\omega$.

Pick $u \in \mathrm{cl}_{\beta Z} D$ and let $X_u = X \cup \{u\} \subseteq \beta X$.

Then $v = \{A \subseteq \omega : u \in \mathrm{cl}_{\beta Z}\{z_n : n \in A\}\} \in \omega^*$.

Let $Y = X \oplus \omega$. Then $Y_v = X \oplus \omega_v$.

Clearly $\omega^2 = X \approx Y$ and $X_u \not\approx Y_v$.

An example

Claim: $X_u \overset{l}{\sim} Y_v$.

Proof:

Note that D is a retract of Y_v and $Y_v \oplus D \approx Y_v$

Then $C_p(Y_v) \overset{l}{\sim} C_{p,D}(Y_v) \times C_p(D) \overset{l}{\sim} C_{p,D}(Y_v)$,

where $C_{p,D}(Y_v) = \{f \in C_p(Y_v) : f(D) = \{0\}\}$.

Define $\phi : C_{p,D}(Y_v) \to C_p(X_u)$ by

$$\phi(f)(x) = \begin{cases} f(v) & \text{if } x = u \\ f(x) + f(n) & \text{if } x \in X_n \text{ for } n < \omega \end{cases}$$

Let $\varepsilon > 0$. There is $V \subset \omega$ s.t. for $n \in V$, $|f(n) - f(v)| < \epsilon/2$.

For $n \in V$ there is $\exists U_n \subseteq X_n$ n.b.h of z_n s.t. $f(U_n) \subseteq (-\epsilon/2, \epsilon/2)$.

An example

We have $U = \bigcup_{n \in V} U_n \cup \{u\}$ is a n.b.h. of $u \in X_u$.

For $x \in U_n$,

$$|\phi(f)(x) - \phi(f)(u)| = |f(x) - f(n) - f(v)| \leq$$

$$|f(x)| + |f(n) - f(v)| < \epsilon.$$

It follows that ϕ is a well-defined continuous linear function.

Define $\psi : C_p(X_u) \to C_{p,D}(Y_v)$ by

$$\psi(g)(y) = \begin{cases} g(u) & \text{if } y = v \\ g(x) - g(z_n) & \text{if } x \in X_n \text{ for } n < \omega \\ g(z_n) & \text{if } x = n \text{ for } n < \omega \end{cases}$$

Then ψ is well-defined, linear and continuous. Moreover $\psi = \phi^{-1}$.

References

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