Ring constructions: axioms needed

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A plain start

AC is equivalent to

1. Zorn's lemma
2. well-ordering principle;
3. Hausforff maximum principle;

   "in any partially ordered set, every totally ordered subset is contained in a maximal totally ordered subset"
4. Tukey's lemma

   "every nonempty collection of finite character has a maximal element with respect to inclusion"

and

AC is equivalent to

on every nonempty set X, there is a

1. a group;
2. an abelian group;
3. a ring;
4. a commutative ring;
5. an integral domain with a unity;

$\vdots$

We are concerned with:

AC is equivalent to

every commutative ring with a unity has a maximal ideal (MIT)

- Krull's observation: AC implies MIT

  Indeed, In a commutative ring R with identity, every proper ideal is contained in a maximal ideal.

- Stone asked whether the converse is true

- Hodges 1978: YES

$$\text{AC} \iff \text{MIT}.$$

Examples of maximal ideals in

- $\mathbb{Z}$
- $\mathbb{Q}$
- $\mathbb{Q}[X]$
- $\mathbb{Q}[[X]]$
- $C[0,1]$

Prime ideals

Theorem

In a commutative ring with identity, each maximal ideal is a prime ideal.

Proof:

Let M be a maximal ideal in a commutative ring with identity 1 and let $xy \in M$. Suppose that $x, y \notin M$. Then, by the maximality of M,

$$M + (x) = R, \qquad M + (y) = R.$$

This gives $a + rx = b + sy = 1$, where $a, b \in M$ and $r, s \in R$.

In particular, we have

$$(a + rx)(b + sy) = 1,$$

$$1 = ab + asy + brx + asxy \in M,$$

a contradiction.

AC needed?

In the integer ring, the zero ideal is not maximal.

There are rings such that all prime ideals are maximal.

Question:
Do we really need to use AC to prove the existence of prime ideals?

- We will come back to this later.

Theorem
Let R be a commutative ring with identity and S be a multiplicatively closed subset of R. Then among the ideals disjoint from S, those maximal elements (existence guaranteed by Zorn's lemma) are prime.

Two motivating theorems

Cohen's Theorem: Noetherian rings

Let R be a commutative ring with identity. If every prime ideal R is finitely generated, then every ideal in the ring is finitely generated, i.e. R is Noetherian.

Issacs' Theorem: PIDs

Let R be an integral domain. If every prime ideal R is principal, then every ideal in R is principal.

Prime ideal principle (Lam and Reyes, 2008)

For suitable ideal families $\mathcal{F}$ in a (commutative) ring, every ideal maximal with respect to not being in $\mathcal{F}$ is prime.

Minimal prime ideals

Given a proper ideal I of R,

- a minimal prime ideal over I is an ideal that is minimal in the set of all prime ideals of R containing I.
- In particular, a minimal prime ideal is a minimal prime ideal over the zero idea.

Minimal prime ideals exist, by Zorn's Lemma (reverse inclusion).

Theorem

For ideals $I \subseteq J$ in a commutative ring R, with J prime, J contains a minimal prime ideal over I.

So, if R is an Artinian ring and M is a maximal ideal in R, then M is minimal.

- All of these are consequences of Zorn's lemma.

How about the converses?

- AC is equivalent to the existence of minimal prime ideals in commutative rings.

Existence of prime ideals: AC not needed

- The existence of prime ideals in commutative rings is equivalent to the Boolean Prime Ideal theorem.

  BPI: ideals in a Boolean algebra can be extended to prime ideals

  PIT: each nontrivial Boolean algebra contains prime ideals

  In Boolean algebras, primes ideals are all maximal.

  SLR: In any ring R, any ideal disjoint from a multiplicatively closed subset S of R is contained in a prime ideal.

- BPI, PIT and SLR are equivalent.

- BPI is weaker than AC (Halpern, 1964).
- The existence of prime ideals in commutative rings with identity is strictly weaker than AC.

Yes, reverse math now

- Friedman, Simpson and Smith (1983):

  1. ACA_0 is equivalent to the statement that every commutative ring with identity contains a maximal ideal.
  2. WKL_0 is equivalent to the statement that every commutative ring with identity contains a prime ideal.

- Hatzikiriakou (1991):

  1. WKL_0 is equivalent to the statement that for any commutative ring R with identity, I a Σ_1 ideal, S a Σ_1 multiplicatively closed set in R with $I \cap S = \emptyset$, there exists a prime ideal P in R containing I and disjoint from S.
  2. ACA_0 is equivalent to the existence of minimal prime ideals in commutative rings.

Related to Noetherian rings

- Simpson (1988): a formal version of Hilbert's basis theorem is equivalent to the statement that ω^ω is well-ordered.
- Hatzikiriakou (1994) considered the reverse of Hilbert's basis theorem for formal power series, same as Simpson's well-orderability of ω^ω.
- Conidis (2019) considered the reverse of the statement that all Artinian rings are Noetherian.
- Sakamoto and Tanaka (2004) considered the reverse of Hilbert's Nullstellensatz.

For homological algebras

Yamazaki (2018): RCA_0 proves the equivalence between

- ACA_0
- Baer's Criterion: "For any ideal J of R and any$R$- homomorphism $g : J \to I$, there exists a R-homomorphism $h : R \to I$ such that $h \restriction J = g$" implies that I is injective.

More projects in this direction.

About radicals

Downey, Lempp and Mileti (2007):

There exist computable commutative rings with identity where the nilradical is Σ_1-complete and the Jocobson radical is Π_2-complete.

Projects on radicals in noncommutative rings.