NATIONAL UNIVERSITY OF SINGAPORE

MA 4207: Mathematical Logic

Semester 2; AY 2019/2020; Midterm Test

Time Allowed: 50 Minutes

INSTRUCTIONS TO CANDIDATES

1. Please write your Student Number. Do not write your name.
2. This assessment paper consists of FIVE (5) questions and comprises ELEVEN (11) printed pages.
3. Students are required to answer **ALL** questions.
4. Students should answer the questions in the space provided.
5. This is a **CLOSED BOOK** assessment.
6. It is permitted to use calculators, provided that all memory and programs are erased prior to the assessment; no other material or devices are permitted.
7. Every question is worth FIVE (5) or SIX (6) marks. The maximum possible marks are 28.

STUDENT NO: ______________________________

This portion is for examiner's use only

Question	Marks	Remarks
Question 1:		
Question 2:		
Question 3:		
Question 4:		
Question 5:		
Total:		

Let $F = \{f : \mathbb{N} \to \mathbb{N}$ such that for all $n \in \mathbb{N}$, $f(n+1) \leq f(n)\}$. How many functions are in F:

- ☐ None; ☐ One; ☐ Two;
- ☐ $\aleph_0$ (that is, countably many);
- ☐ $2^{\aleph_0}$ (that is, as many as real numbers);
- ☐ Any other number of functions.

Explain the answer (the explanations carry the majority of the marks).

Solution. The solution follows the proof of a student, as that proof was better than the planned master solution.

The correct answer is the following: ☒ $\aleph_0$ (that is, countably many).

The reason is as follows. Let $p_0, p_1, p_2, \ldots$ be the list of all prime numbers and

$$g(f) = p_0^{f(0)} \cdot \prod_{k>0} p_k^{f(k-1)-f(k)}.$$

As f can go down from $f(k-1)$ to $f(k)$ at most $f(0)$ times and f never goes properly up, there are only finitely many factors in this product which are not 1 and these factors are natural numbers and therefore the product $g(f)$ is well defined and is a natural number. Furthermore, one can reconstruct f from $g(f)$, as the number of p_0 in the prime factorisation gives the value of $f(0)$ and, for every $k > 0$, the number of p_k in the prime factorisation gives the difference $f(k-1) - f(k)$, so that one can define inductively all values $f(k)$ from $g(f)$. Thus g has an inverse, g is one-one and $\text{Card}(F) \leq \aleph_0$.

Furthermore, $\aleph_0 \leq \text{Card}(F)$, as for every natural number ℓ there is the constant function f_ℓ with $f_\ell(k) = \ell$ for all k. The Schröder-Bernstein Theorem says now that if the lower bound and upper bound of a cardinal are the same, say are both $\aleph_0$, then the given cardinal must be $\aleph_0$. In other words, the cardinals are linearly ordered. Thus $\text{Card}(F) = \aleph_0$.

In this and further questions, let $\wedge$ denote "and", $\vee$ denote "inclusive or", $\neg$ denote "not", $\rightarrow$ denote "implies", $\leftrightarrow$ denote "logically equal" and $\oplus$ denote "exclusive or".

Let A_1, A_2, A_3, A_4 be the first four atoms and $b = A_1 + 2 \cdot A_2 + 4 \cdot A_3 + 8 \cdot A_4$ be a number from 0 to 15 obtained when interpreting $A_4A_3A_2A_1$ as a binary number. Provide a formula α which outputs 1 iff b is the remainder of a square number divided by 13. For example, if $b = 9$ or $b = 10$ then α should be 1, as 9 is the remainder of 3^2 divided by 13 and 10 is the remainder of 7^2 divided by 13. However, if $b = 8$ then α should be 0 as 8 is not the remainder of any square number divided by 13. For the values $b = 13, 14, 15$, as they are not the remainder of any number divided by 13, the output should be 0.

Solution. Note that, modulo 13, $h^2 = (13 - h)^2$, so to capture all squares modulo 13, one only needs to compute the squares of $0, 1, 2, 3, 4, 5, 6$; these are modulo 13 the numbers $0, 1, 4, 9, 3, 12, 10$, respectively. So α returns 1 at $b = 0, 1, 3, 4, 9, 10, 12$ and 0 at $b = 2, 5, 6, 7, 8, 11, 13, 14, 15$. A possible formula for α is the following:

$$\begin{aligned}\alpha \;=\; & (\neg A_4 \wedge \neg A_3 \wedge (\neg(A_2) \vee A_1)) \;\vee \\ & (A_3 \wedge \neg A_2 \wedge \neg A_1) \;\vee \\ & (A_4 \wedge \neg A_3 \wedge (A_2 \oplus A_1)).\end{aligned}$$

This formula is 1 iff $b \in \{0, 1, 3, 4, 9, 10, 12\}$.

Let $\leftrightarrow, \oplus, \wedge, \vee$ be logical connectives and $0, 1/2, 1$ be truth-constants. Recall the following rules to evaluate connectives in $\{0, 1/2, 1\}$-valued fuzzy logic: $r \wedge s = \min\{r, s\}$, $r \vee s = \max\{r, s\}$, $r \oplus s = \min\{r+s, 2-r-s\}$, $r \leftrightarrow s = \min\{1+r-s, 1+s-r\}$. Construct a formula with two atoms A_1, A_2 using above connectives and truth-constants such that $B^2_\alpha(x, y)$ is defined as follows:

$$B^2_\alpha(x,y) = \begin{cases} 1 & \text{if } x = 1 \text{ and } y = 1; \\ 1/2 & \text{if } x = 1/2 \text{ and } y = 1/2; \\ 0 & \text{otherwise.} \end{cases}$$

Explain how one can generalise this construction to find for any given function f from $\{0, 1/2, 1\}^n$ to $\{0, 1/2, 1\}$ a formula α such that $B_\alpha = f$ when α is interpreted in the given $\{0, 1/2, 1\}$-valued fuzzy logic.

Solution. The first idea is that one can compare the value of an atom A_k with a truth-constant. The following table gives expressions which compare whether an item β, say an atom or negated atom or any other expression, take the corresponding truth-value, the expression returns 1 if yes and returns 0 if not:

- Value 1: $\beta \wedge ((\beta \oplus \beta) \oplus 1)$;
- Value 1/2: $\beta \oplus \beta$;
- Value 0: $(\beta \oplus 1) \wedge ((\beta \oplus \beta) \oplus 1)$.

This allows to construct the formula as follows:

$$\begin{aligned} \alpha \;=\;& (A_1 \wedge ((A_1 \oplus A_1) \oplus 1) \wedge A_2 \wedge ((A_2 \oplus A_2) \oplus 1) \wedge 1) \;\vee \\ & ((A_1 \oplus A_1) \wedge (A_2 \oplus A_2) \wedge 1/2). \end{aligned}$$

Note that negated atoms A_k are just $(A_k \oplus 1)$ and that the above list allows to compare the value of each A_k with any given truth-value and then to connect by an "and" all the terms for each row in the truth-table and the resulting conjunction is then connected with an "and" to the logical constant which is the value of this row; afterwards, all the so created row-terms are connected by "or". As in the formula α above, one can omit row-terms which map to constant 0 in this disjunction. With this method one can create the fuzzy equivalent of a disjunctive normal form expression in the three-valued fuzzy logic and match the resulting function even exactly at all values.

Let $C(\alpha, \beta)$ connect α and β with one of the connectives $\wedge$, $\vee$, $\oplus$, $\rightarrow$, $\leftrightarrow$ and consider the set $S_C = \{C(A_i, A_j) : i \neq j\}$ where i, j run over all possible indices of the atoms; these indices are $1, 2, 3, \ldots$ and there are countably many atoms (that is, $\aleph_0$ many).

For each of the possible sets $S_\wedge, S_\vee, S_\oplus, S_\rightarrow, S_\leftrightarrow$, determine how many truth-assignments ν make all formulas in the set true.

Solution. And ($\wedge$): Here all atoms have to be true, so there is exactly one truth-assignment.

Or ($\vee$): Here all atoms except at most one have to be true. As one can select each atom to be the unique one which is 0, there are countably many truth-assignments.

Exclusive or ($\oplus$): If A_i and A_j have the same truth-value, then $A_i \oplus A_j$ is false. Furthermore, there are infinitely many atoms, so at least three and thus two of them have the same truth-value, thus there is no truth-assignment which makes S true.

Implication ($\rightarrow$): If A_i is 1 so also all A_j with $j \neq i$. Furthermore, all atoms can be 0. So there are two truth-assignments, that one where all A_i are 0 and that one where all A_i are 1.

Equivalence ($\leftrightarrow$): Now all atoms A_i and A_j need to have the same truth-value. Thus there are exactly two truth-assignments which make all formulas in S true; the one which maps all atoms to 0 and the one which maps all atoms to 1.

The company Traffic Innovation Ltd wants to develop and manufacture four-coloured traffic lights. In order to know whether this product would have a chance, it asked at the Ministry of Motorisation and Traffic Control of its country for a set of rules which these four-coloured traffic lights would have to follow. The ministry provided the following rules which regulate the permitted colours and the possible values of a function f which tells how the colours follow one after the other while the traffic light operates:

1. $blue, red, green, yellow \in Colour$;
2. $\forall x, y \in Colour\, [f(x) = f(y) \rightarrow x = y]$;
3. $\forall x \in Colour\, [f(x) \neq x]$;
4. $f(green) = yellow$;
5. $f(f(green)) \neq green$.

(a) How many functions f to define the successor-colours of a colour are permitted by these rules for the set $Colour = \{blue, red, green, yellow\}$? List all these functions.

(b) The marketing chief of Traffic Innovation Ltd got the idea to check whether one could also find an f respecting all rules for a set with five or more colours. Furthermore, he made the additional rule that such an f should go through all the colours in one cycle. He proposed the set $Colour = \{blue, red, green, yellow, white\}$. How many possible cycles are there going through all five colours respecting all the rules above? List one of them.

(c) Furthermore, find the smallest number of colours where there exists an f satisfying all rules such that f is not a single cycle and list a witnessing f.

Solution. For the ease of notation, one uses th arrow-notation "`green -> yellow`" instead of "`f(green)=yellow`" and so on in the following diagrammes.

For (a) f makes only four-cycles. The following two are possible:

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green -> yellow -> red -> blue -> green;
green -> yellow -> blue -> red -> green.
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For (b), f can make six five-cycles by putting green first, then yellow and then the three remaining colours in any order:

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green -> yellow -> white -> red -> blue -> green;
green -> yellow -> white -> blue -> red -> green;
green -> yellow -> blue -> red -> white -> green;
green -> yellow -> blue -> white -> red -> green;
green -> yellow -> red -> blue -> white -> green;
green -> yellow -> red -> white -> blue -> green.
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For (c), the smallest size is five and one has to make a three-cycle and a two-cycle. The three-cycle must contain the colours yellow and green plus a third one from the other colours. The two-cycle consists of the two remaining colours. Here an example:

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green -> yellow -> red -> green, white -> blue -> white.
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END OF PAPER