NATIONAL UNIVERSITY OF SINGAPORE

MA4207 - MATHEMATICAL LOGIC

(Semester 2 : AY2020/2021)

Time allowed : 150 minutes

INSTRUCTIONS TO CANDIDATES

1. Submit a signed version of the honours declaration to the corresponding Luminus Folder (if not already done so).
2. Use A4 paper and pen (blue or black ink) to write your answers. Write on one side of the paper only. Write the question number and page number on the top right corner of each page. Write your student number clearly on the top left of every page of the exam. Do not write your name.
3. This examination paper contains TEN (10) questions and comprises ELEVEN (11) pages. Answer **ALL** questions.
4. The total mark for this paper is SIXTY (60).
5. This is an **OPEN BOOK** exam. You can use the lecture notes and the slides for the exam by viewing them on the computer; the screen of the computer must be recorded.
6. You may use any calculator. However, you should lay out systematically the various steps in the calculation.
7. Join the Zoom conference and turn on the video setting at all time during the exam. Adjust the camera such that your face and upper body including your hands are captured on Zoom.
8. You may go for a short toilet break (not more than 5 minutes) during the exam.
9. At the end of the exam,
   - scan or take pictures of your work (make sure the images can be read clearly) together with the declaration form;
   - merge all your images into one pdf file (arrange them in the order: Question 1, Question 2, ..., Question 10);
   - name the file by student number (a.k.a. matric number) followed by hyphen followed by course-code MA4207, for example, `A1283125Z-MA4207.pdf`.
   - upload your pdf into the LumiNUS folder "Exam Submission".
   - **After** uploading your exam, stop the screen recording and upload it in the corresponding folder.
10. The folder "Exam Submission" will close at 11:50 hrs; no submission will be accepted afterwards, unless there is a valid reason.

Question 1 [6 Marks]
Construct a formula α using the atoms $A_1, A_2, A_3, A_4, A_5, A_6$ which outputs 1 for square numbers and 0 for prime numbers and does not need to satisfy any requirements for other inputs. Here the bitsequence $A_1A_2A_3A_4A_5A_6$ of the values of the atoms is interpreted as the number $32 \cdot A_1 + 16 \cdot A_2 + 8 \cdot A_3 + 4 \cdot A_4 + 2 \cdot A_5 + A_6$, so 001001 is nine, a square number. Up to seven of the Boolean connectives $\wedge$ (and), $\vee$ (or), $\oplus$ (exclusive or) are allowed, furthermore, arbitrarily many $\neg$ (negations) and brackets.

Solution. One possible solution is

$$\neg A_5 \wedge (\neg A_6 \vee (A_6 \wedge \neg A_4 \wedge ((A_1 \oplus A_3) \vee (\neg A_2 \oplus A_3)))),$$

The square numbers in question are, in binary

$$000000, 000001, 000100, 001001, 010000, 011001, 100100, 110001.$$

Note that the odd square numbers end with 001 and the even ones with 00. So the main conflict are prime numbers ending with 001 and these are 010001 (seventeen) and 101001 (forty-one). So the formula would be to give 1 when the binary number either ends with 00 or when it with 001 and does not start neither with 010 nor 101. For the end condition, one can move out $\neg A_5$ in order to enforce that the second last digit is always 0. If the last digit is 0, no further check is needed, if the last digit is 1, one requires also that A_4 is 0 and either A_1 differs from A_3 or A_2 equals A_3 in order to get the desired formula. The resulting formula is displayed above. So one computes B^6_α for the displayed formula α above.

Question 2 [6 Marks]
Determine for each of the following sets of formulas say whether they are unsatisfiable, satisfiable but not all tautologies, entirely consisting of tautologies; give reasons for each decision:

1. $\{(A_1 \vee A_2) \rightarrow A_1, (A_1 \wedge A_2) \rightarrow A_2\}$;
2. $\{(A_1 \oplus A_2) \vee (A_1 \leftrightarrow A_2)\}$;
3. $\{A_1 \rightarrow A_2, A_2 \rightarrow A_3, A_3 \rightarrow A_4, A_4 \rightarrow A_1, A_1 \oplus A_2 \oplus A_3 \oplus A_4\}$.

Solution. 1. The first formula is not a tautology, as one sees when atom A_1 takes the truth-value 0 and A_2 takes the truth-value 1. However, the set of formulas is satisfiable, as one sees when all two atoms are 1.

2. $A_1 \oplus A_2$ is 1 iff $A_1 \leftrightarrow A_2$ is 0; thus the disjunction of these two terms is always satisfied. So this set of formulas consists entirely of one tautology.

3. The first four implications form some type of circle, therefore either all four atoms are 1 or all four atoms are 0. The last formula is, however, only satisfied when an odd number of atoms is 1. Thus they cannot be all four 1 and they also cannot be all four 0. Hence the set of formulas is unsatisfiable.

Question 3 [6 Marks]
Let a formula α be in a set C of formulas if it only uses the connectives $\vee$ (defined as maximum) $\wedge$ (defined as minimum) and $\neg$ (mapping r to $1-r$) but no other connectives and no truth-constants. Allow the Fuzzy truth-values $Q = \{0.0, 0.2, 0.4, 0.6, 0.8, 1.0\}$. Prove by structural induction the following: All formulas $\alpha \in C$ satisfy that if ν takes on all atoms truth-values from $\{0.2, 0.4, 0.6, 0.8\}$ then $\overline{\nu}(\alpha) \in \{0.2, 0.4, 0.6, 0.8\}$.

Solution. Let $P = \{0.2, 0.4, 0.6, 0.8\}$. By induction one shows the following statement: If ν maps all atoms to P then $\overline{\nu}(\alpha) \in P$.

Now consider any α and assume that by induction hypothesis, the assumption holds for β, γ whenever α is obtained from β, γ using one single connective (this assumption is void in the case that α is an atom). Note that one needs only to show this for connectives which are used for building formulas in C.

If α is an atom A_k then $\overline{\nu}(A_k) = \nu(A_k)$ by definition and $\overline{\nu}(A_k) \in P$ by choice of ν.

Now assume that $\alpha \in C$ and α is $\neg\beta$. By induction hypothesis, $\overline{\nu}(\beta) \in \{0.2, 0.4, 0.6, 0.8\}$ and $\overline{\nu}(\alpha) = 1 - \overline{\nu}(\beta)$. One sees easily that $\overline{\nu}(\alpha)$ is one of the values $0.8 = 1 - 0.2, 0.6 = 1 - 0.4, 0.4 = 1 - 0.6, 0.2 = 1 - 0.8$ and also in P.

Now assume that $\alpha \in C$ and α is either $\beta \vee \gamma$ or $\beta \wedge \gamma$ for $\beta, \gamma \in C$. By induction hypothesis, β, γ satisfy $\overline{\nu}(\beta) \in P$, $\overline{\nu}(\gamma) \in P$. Now $\overline{\nu}(\alpha)$ is either the minimum or the maximum of the two values $\overline{\nu}(\beta)$ and $\overline{\nu}(\gamma)$ which are both in P. Thus $\overline{\nu}(\alpha) \in P$. This concludes the inductive step and the proof.

Note that for the choice of C the selection of the connectives was crucial, as $0.2 \leftrightarrow 0.2$ and $0.2 \oplus 0.8$ both evaluate to 1 which is outside P.

Question 4 [6 Marks]
Assume that a logical language contains predicates R, B, G, Y, W for things having the colours red, blue, green, yellow and white, respectively. So $W(x)$ says that the variable x denotes a thing which has (perhaps among other colours) the colour white. Formalise the following verbal statements in logic:

1. There is a thing which has the colour red and there is a thing which has the colour blue;
2. All green things have also the colour yellow;
3. Exactly one thing has both colours, yellow and white;
4. No thing has all of the colours red, blue and green;
5. There are at least two things which have both the colours green and white;
6. Every thing which has both colours yellow and red has also the colour blue as well.

Solution. The formulas are as follows:

1. $\exists x\, \exists y\, [R(x) \wedge B(y)]$;
2. $\forall x\, [G(x) \rightarrow Y(x)]$;
3. $\exists x\, \forall y\, [Y(x) \wedge W(x) \wedge ((Y(y) \wedge W(y)) \rightarrow x = y)]$;
4. $\forall x\, [\neg R(x) \vee \neg B(x) \vee \neg G(x)]$;
5. $\exists x\, \exists y\, [x \neq y \wedge G(x) \wedge G(y) \wedge W(x) \wedge W(y)]$;
6. $\forall x\, [(Y(x) \wedge R(x)) \rightarrow B(x)]$.

Question 5 [6 Marks]
Consider the additive structure of the natural numbers, $(\mathbb{N}, 0, +)$ where $0 \in \mathbb{N}$. Give formulas (with free variables x and, if applicable, y) which are equivalent in this logic to the following verbal statements (where equality $=$ and its negation $\neq$ are available, but no constants besides 0):

1. There is at most one odd number less than or equal to x and x is not 0;
2. $y \geq x + 2$;
3. x is of the form $3y$ or $5y$ and also odd.

Write a few words explaining each formula and note that $<, \leq, >, \geq$ are not available in the logical language, so quantified formulas have to replace them.

Solution. 1. $\forall u \, \forall v \, \forall w \, [((u \neq 0 \wedge v \neq 0 \wedge w \neq 0) \rightarrow x \neq u + v + w) \wedge x \neq 0]$. So the first two odd numbers are 1 and 3 and all numbers from 3 onwards cannot be x; the numbers from 3 onwards are the sums of three nonzero natural numbers. So x has to be different from those and different from 0. There are no further conditions.

2. $\exists v \, \exists w \, [v \neq 0 \wedge w \neq 0 \wedge y = x + v + w]$. So y is the sum of x and two nonzero natural numbers.

3. $(x = y + y + y \vee x = y + y + y + y + y) \wedge \forall u \, [x \neq u + u]$. The first two conditions relate x and y, so they do not need a quantifier. The last condition says that x is odd. This one can express by either saying that x is of the form $u + u + 1$ or by saying that x cannot be of the form $u + u$. The latter is used, as the constant 1 is not available.

Question 6 [6 Marks]
Assume that the following axioms are given where $+$ is a binary operation:

1. $\forall x \, \forall y \, \forall z \, [(x+y)+z = x+(y+z)]$;
2. $\forall x \, \forall y \, [x+x+x+x = y+y+y+y]$;
3. $\forall x \, \forall y \, [x+y = y+x]$;
4. $\exists x \, \exists y \, [x+x = y+y \wedge x \neq y \wedge x \neq y+y+y \wedge x+x+x \neq y \wedge x+x+x \neq y+y+y \wedge x+x \neq y+y+y+y \wedge x+x+x+x \neq y+y]$.

Provide a finite structure (semigroup) which satisfies all axioms. Explain why the structure works.

Solution. There are many correct solutions. Here two examples.

(a) Let $H = \{0,1,2,3\}$, $+$ be addition modulo 4 and $G = H \times H$ with coordinatewise addition modulo 4. Then $x+x+x+x = (0,0)$ for all x, as in both coordinates, four times adding a number with itself gives 0. Furthermore, addition in G is associative and commutative. For the last one, one takes $x = (1,3)$ and $y = (1,1)$. Then $y+y+y = (3,3)$ and $x+x+x = (3,1)$, so these four elements of G are pairwise distinct. Furthermore, $x+x = y+y = (2,2)$ and $x+x+x+x$ and $y+y+y+y$ are both $(0,0)$, so $x+x$ and $y+y$ are both distinct from the neutral element $(0,0)$. Thus all the conditions of the last axiom are also satsified. Furthermore, as the structure has 16 elements, it is finite. The variant of (a) where one has the product of H and a two-element group also works. However, H alone (instead of G) does not work.

(b) Here one has $\{1,2,3,4\} \times \{0,1\} - \{(4,1)\}$ where $(a,b)+(c,d) = (a+c, b \oplus d)$ where $\oplus$ is exclusive or in the case that $a+c < 4$ and $(a,b)+(c,d) = (4,0)$ in the case that $a+c \geq 4$. Thus the sum of four elements is always $(4,0)$. Furthermore, one sees that this operation is commutative. The law of associativity needs a case-distinction. When adding three numbers and the first coordinate of at least one is greater than or equal to 2 then the result is $(4,0)$, thus associativity holds in this case. Furthermore, $(1,c)+(1,d)+(1,e)$ is, independently of putting the brackets, $(3, c \oplus d \oplus e)$ what is associative due to the associativity of the exclusive or. Now choosing $x = (1,0)$ and $y = (1,1)$ satisfies the fourth axiom, as $x+x = y+y = (2,0)$ and $x+x+x = (3,0)$ and $y+y+y = (3,1)$.

Question 7 [6 Marks]
Let a logical language contain the equality $=$ and three constants $0, 1, 2$. Furthermore consider the model $\mathfrak{A} = (\mathbb{N}, 0, 1, 2)$ where the domain is the natural numbers and $0, 1, 2$ refer to the usual numbers; however, the successor-function, $+$ and $\cdot$ are not in the logical language and therefore not part of the model.

Let T be the theory of all sentences in the given logical language which are true in the model $\mathfrak{A}$.

Recall that the theory T is κ-categorical iff it contains a model of size κ and if all models of size κ are isomorphic.

Provide a recursively enumerable set S of sentences such that $S \models \alpha$ for all $\alpha \in T$ and say for each $\kappa \in \{5, \aleph_0, \aleph_1\}$ whether theory is κ-categorical and give reasons for the answer.

Solution. S contains the axioms $0 \neq 1$, $0 \neq 2$, $1 \neq 2$ plus for each number n an axiom α_n which says that there are more than n elements,

$$\alpha_n = \forall x_1 \forall x_2 \ldots \forall x_n \exists y \, [y \neq x_1 \wedge y \neq x_2 \wedge \ldots \wedge y \neq x_n].$$

T must contain every α_n as the model $\mathfrak{A}$ is infinite. Furthermore, in the model $\mathfrak{A}$ are the three constants different, so this must also be added into S.

Furthermore, the models of $\{0 \neq 1, 0 \neq 2, 1 \neq 2\}$ are just all sets with at least three elements where the three constants are different; between two such sets of the same cardinality, one can make a bijection which maps the representatives of each constant $c \in \{0, 1, 2\}$ in one model to that representative of the same constant in the other model. Thus these models are all κ-categorical for all $\kappa \geq 3$. However, the α_n exclude all finite models, thus the resulting theory T generated by the full S is $\aleph_0$-categorical and $\aleph_1$-categorical, but not 5-categorical.

Question 8 [6 Marks]
Let c, d be constants and P be a uniary predicate. Find consistent sets S, T, U of sentences not containing the constants c, d plus the corresponding proofs for the below or say that the sets of sentences and corresponding proofs cannot exist and explain why.

1. $S \vdash c = d$;
2. $T \vdash P(c), T \vdash P(d)$;
3. $U \vdash P(c), U \vdash \neg P(d)$.

Solution. 1. One selects $S = \{\forall x \, \forall y \, [x = y]\}$. This formula is consistent, as there is a model with exactly one element. Now applying Axiom 2 twice onto this formula with modus ponens allows to deduce that $c = d$:

1. $S \vdash \forall x \, \forall y \, [x = y]$ (Copy);
2. $S \vdash \forall y \, [c = y]$ (Axiom 2, $x \to c$, Modus Ponens);
3. $S \vdash c = d$ (Axiom 2, $y \to d$, Modus Ponens).

2. One selects $T = \{\forall x \, [P(x)]\}$. Again the formulas $P(c)$ and $P(d)$ can be proven easily from T by the following proof:

1. $T \vdash \forall x \, [P(x)]$ (Copy);
2. $T \vdash P(c)$ (Axiom 2, $x \to c$, Modus Ponens);
3. $T \vdash P(d)$ (Axiom 2, $x \to d$, Modus Ponens).

3. A consistent set U with this property does not exist. For this recall that by the Principle of Generalisation of Constants, if the set U can prove α_c^z for a constant c neither occurring in α nor in U then $U \vdash \forall z \, [\alpha]$. So let c, d not occur at all in U. Now one has the following formal proof:

1. $U \vdash P(c)$ (Assumption on c);
2. $U \vdash \forall z \, [P(z)]$ (Generalisation of Constants);
3. $U \vdash P(d)$ (Axiom 2, $z \to d$, Modus Ponens);
4. $U \vdash \neg P(d)$ (Assumption on d).

The last two steps show that U is inconsistent. Thus a consistent U which does not contain the constants c and d cannot prove both, $P(c)$ and $\neg P(d)$.

Question 9 [6 Marks]
Make a formal proof that the following statement is valid:

$$\forall x\,[f(f(x)) = x] \to \forall y\,[y = f(f(y))].$$

This statement should be proven directly from Λ, the Deduction Theorem and the Generalisation Theorem, but without using the Principle of Alphabetic Variants.

Solution. The derivation is as follows:

1. $\emptyset \vdash \forall x\,[f(f(x)) = x] \to f(f(y)) = y$ (Axiom 2);
2. $\{\forall x\,[f(f(x)) = x]\} \vdash f(f(y)) = y$ (Deduction Theorem);
3. $\{\forall x\,[f(f(x)) = x]\} \vdash v = w \to v = v \to w = v$ (Axiom 6);
4. $\{\forall x\,[f(f(x)) = x], v = w\} \vdash v = v \to w = v$ (Deduction Theorem);
5. $\{\forall x\,[f(f(x)) = x], v = w\} \vdash v = v$ (Axiom 5);
6. $\{\forall x\,[f(f(x)) = x], v = w\} \vdash w = v$ (Modus Ponens);
7. $\{\forall x\,[f(f(x)) = x]\} \vdash v = w \to w = v$ (Deduction Theorem);
8. $\{\forall x\,[f(f(x)) = x]\} \vdash \forall w\,[v = w \to w = v]$ (Generalisation Theorem);
9. $\{\forall x\,[f(f(x)) = x]\} \vdash \forall v\,\forall w\,[v = w \to w = v]$ (Generalisation Theorem);
10. $\{\forall x\,[f(f(x)) = x]\} \vdash \forall w\,[f(f(y)) = w \to w = f(f(y))]$ (Axiom 2, $v \to f(f(y))$, Modus Ponens);
11. $\{\forall x\,[f(f(x)) = x]\} \vdash f(f(y)) = y \to y = f(f(y))$ (Axiom 2, $w \to y$, Modus Ponens);
12. $\{\forall x\,[f(f(x)) = x]\} \vdash y = f(f(y))$ (Modus Ponens);
13. $\{\forall x\,[f(f(x)) = x]\} \vdash \forall y\,[y = f(f(y))]$ (Generalisation Theorem);
14. $\emptyset \vdash \forall x\,[f(f(x)) = x] \to \forall y\,[y = f(f(y))]$ (Deduction Theorem).

Question 10 [6 Marks]
Is the below statement valid? If so, prove it formally using the axioms from Λ and the Deduction Theorem and the Generalisation Theorem, if not provide a model where it is, for some default of the variables, false. The predicate P in the statement is unary, that is, depends only on one term. The statement is this:

$$P(x) \to \forall y\,[P(y) \to P(x)].$$

Solution. The formula is valid. One can see it like this: If $P(x)$ is true then $P(y) \to P(x)$ is true for all y, if $P(x)$ is false then the overall implication is true independently of what happens to the right side of the first implication. A formal proof for this statement goes as follows:

1. $\emptyset \vdash P(x) \to P(y) \to P(x)$ (Axiom 1, as $A \to B \to A$ is a tautology independently of the values of the atoms A and B);
2. $\{P(x)\} \vdash P(y) \to P(x)$ (Deduction Theorem);
3. $\{P(x)\} \vdash \forall y\,[P(y) \to P(x)]$ (Generalisation Theorem, y does not occur free in the preconditions);
4. $\emptyset \vdash P(x) \to \forall y\,[P(y) \to P(x)]$ (Deduction Theorem).