Covering the recursive sets*

Bjørn Kjos-Hanssen†, Frank Stephan‡ and Sebastiaan A. Terwijn§

11 August 2016

Abstract

We give solutions to two of the questions in a paper by Brendle, Brooke-Taylor, Ng and Nies. Our examples derive from a 2014 construction by Khan and Miller as well as new direct constructions using martingales.

At the same time, we introduce the concept of i.o. subuniformity and relate this concept to recursive measure theory. We prove that there are classes closed downwards under Turing reducibility that have recursive measure zero and that are not i.o. subuniform. This shows that there are examples of classes that cannot be covered with methods other than probabilistic ones. It is easily seen that every set of hyperimmune degree can cover the recursive sets. We prove that there are both examples of hyperimmune-free degree that can and that cannot compute such a cover.

1 Introduction

An important theme in set theory has been the study of cardinal characteristics. As it turns out, in the study of these there are certain analogies with recursion theory, where the recursive sets correspond to sets in the ground model. Recently, Brendle, Brooke-Taylor, Ng and Nies [1] point out analogies between cardinal characteristics and the study of algorithmic randomness. We address two questions raised in this paper that are connected to computing covers for the recursive sets.

*This This work was partially supported by a grant from the Simons Foundation (Grant number 315188 to Bjørn Kjos-Hanssen), the National Science Foundation (Grant number NSF-DMS-1545707 to Bjørn Kjos-Hanssen) and by a grant from the Ministry of Education in Singapore (number MOE2013-T2-1-062 / R146-000-184-112 to F. Stephan). A substantial part of the work was performed while the first and third authors were supported by the Institute for Mathematical Sciences of the National University of Singapore during the workshop on *Algorithmic Randomness* from 2 to 30 June 2014.

†Department of Mathematics, University of Hawaii at Manoa, Honolulu, Hawaii 96822, USA, bjoern.kjos-hanssen@hawaii.edu.

‡Department of Mathematics and Department of Computer Science, National University of Singapore, 10 Lower Kent Ridge Road, Block S17, Singapore 119076, Republic of Singapore, fstephan@comp.nus.edu.sg.

§Department of Mathematics, Radboud University Nijmegen, P.O. Box 9010, 6500 GL Nijmegen, The Netherlands, terwijn@math.ru.nl.

In the following, we will assume that the reader is familiar with various notions from computable measure theory, in particular, with the notions of Martin-Löf null, Schnorr null and Kurtz null set. For background on these notions we refer the reader to the books of Calude [2], Downey and Hirschfeldt [6], Li and Vitányi [14] and Nies [16].

Our notation from recursion theory is mostly standard, except for the following: The natural numbers are denoted by ω, 2^ω denotes the Cantor space and $2^{<\omega}$ the set of all finite binary sequences. $\mathbb{R}^{\geqslant 0}$ denotes the set of those real numbers which are not negative. We denote the concatenation of strings σ and τ by $\sigma\tau$. The notation $\sigma \sqsubseteq \tau$ denotes that the finite string σ is an initial segment of the (finite or infinite) string τ. We identify sets $A \subseteq \omega$ with their characteristic sequences, and $A \upharpoonright n$ denotes the initial segment $A(0) \dots A(n-1)$. We use λ to denote the empty string. Throughout, μ denotes the Lebesgue measure on 2^ω. We write $a \simeq b$ if either both sides are undefined, or they are both defined and equal. We let $\mathrm{Parity}(x) = 0$ if x is even, and $\mathrm{Parity}(x) = 1$ if x is odd.

Definition 1. A function $M : 2^{<\omega} \to \mathbb{R}^{\geqslant 0}$ is a *martingale* if for every $x \in 2^{<\omega}$, M satisfies the averaging condition

$$2M(\sigma) = M(\sigma 0) + M(\sigma 1),$$

A martingale M *succeeds on* a set A if

$$\limsup_{n\to\infty} M(A \upharpoonright n) = \infty.$$

The class of all sets on which M succeeds is denoted by $S[M]$.

For more background material on recursive martingales we refer the reader to the above mentioned textbooks [2, 6, 14, 16]. The following definition is taken from Rupprecht [19, 20].

Definition 2. An oracle A is *Schnorr covering* if the union of all Schnorr null sets is Schnorr null relative to A. An oracle A is *weakly Schnorr covering* if the set of recursive reals is Schnorr null relative to A. For the latter, we will also say that A Schnorr covers REC.

Definition 3. A *Kurtz test* relative to A is an A-recursive sequence of closed-open sets G_i such that each G_i has measure at most 2^{-i}; these closed-open sets are given by explicit finite lists of strings and they consist of all members of $\{0,1\}^\omega$ extending one of the strings. Note that $i \to \mu(G_i)$ can be computed relative to A. The intersection of a Kurtz test (relative to A) is called a Kurtz null set (relative to A). An oracle A is *Kurtz covering* if there is an A-recursive array $G_{i,j}$ of closed-open sets such that each i-th component is a Kurtz test relative to A and every unrelativised Kurtz test describes a null-set contained in $\cap_j G_{i,j}$ for some i; A is *weakly Kurtz covering* if there is such an array and each recursive sequence is contained in some A-recursive Kurtz null set $\cap_j G_{i,j}$.

Brendle, Brook-Taylor, Ng and Nies [1] called the notion of (weakly) Schnorr covering in their paper (weakly) Schnorr *engulfing*. In this paper, we will use the original terminology of Rupprecht [19, 20]. We have analogous notions for the other notions of effective null sets. As mentioned above, a set A is weakly Kurtz covering if the set of recursive reals is Kurtz null relative to A. We also have Baire category analogues of these notions of covering: A set A is *weakly meager covering* if it computes a meager set that contains all recursive reals; more precisely, A is weakly meager covering iff there is an A-recursive function f mapping each binary string σ to an extension $f(\sigma)$ such that every recursive sequence B has only finitely many prefixes σ for which $f(\sigma)$ is also a prefix of B. Recall that a set A is diagonally nonrecursive (DNR) if there is a function $f \leqslant_T A$ such that, for all x, if $\varphi_x(x)$ is defined then $\varphi_x(x) \neq f(x)$. A set A has hyperimmune-free Turing degree if for every $f \leqslant_T A$ there is a recursive function g with $\forall x\, [f(x) \leqslant g(x)]$.

2 Solutions to Open Problems

Brendle, Brooke-Taylor, Ng and Nies [1, Question 4.1], posed three questions, (7), (8) and (9). In this section, we will provide the answers to the questions (7) and (9). For this we note that by [1, Theorem 4.2] / [20, Corollary VI.12] and [11, Theorem 5.1], we have the following result.

Theorem 4. *A set A is weakly meager covering iff it is high or of* DNR *degree.*

We recall the following well-known definitions and results.

Definition 5. A function ψ, written $e \mapsto (n \mapsto \psi_e(n))$, is a recursive numbering if the function $(e, n) \mapsto \psi_e(n)$ is partial recursive. For a given recursive numbering ψ and a function h, we say that f is DNR_h^ψ if for all n, $f(n) \neq \psi_n(n)$ and $f(n) \leq h(n)$. An *order function* is a recursive, nondecreasing, unbounded function.

Theorem 6 (Khan and Miller [10, Theorem 4.3])**.** *For each recursive numbering ψ and for each order function h, there is an $f \in \mathrm{DNR}_h^\psi$ such that f computes no Kurtz random real.*

Wang (see [6, Theorem 7.2.13]) gave a martingale characterisation of Kurtz randomness. While it is obvious that weakly Kurtz covering implies weakly Schnorr covering for the martingale notions, some proof is needed in the case that one uses tests (as done here).

Proposition 7. *If A is weakly Kurtz covering then A is weakly Schnorr covering.*

Proof. Suppose A is weakly Kurtz covering, as witnessed by the A-recursive array of closed-open sets $G_{i,j}$. Then the sets $F_j = \cup_i G_{i,i+j+1}$ form an A-recursive Schnorr test, as each F_j has at most the measure $\sum_j 2^{-i-j-2} = 2^{-i-1}$ and the measures of the F_j is uniformly A-recursive as one can relative to A

compute the measure of each $G_{i,i+j+1}$ and their sum is fast converging. As for each recursive set there is an i such that all $G_{i,i+j+1}$ contain the set, each recursive set is covered by the Schnorr test. $\square$

Theorem 8. *There is a recursive numbering ψ and an order function h such that for each set A, if A computes a function f that is DNR_h^ψ then A is weakly Kurtz covering.*

Proof. Fix a correspondence between strings and natural numbers num : $2^{<\omega} \to \omega$ such that

$$2^{|\sigma|} - 1 \leq \mathrm{num}(\sigma) \leq 2^{|\sigma|+1} - 2.$$

For instance, $\mathrm{num}(\sigma)$ could be the position of σ in the length-lexicographically lexicographic ordering of all strings as proposed by Li and Vitányi [14]. Let $\mathrm{str}(n) = \mathrm{num}^{-1}(n)$ be the string representation of the number n. Thus

$$2^{|\mathrm{str}(n)|} - 1 \leq \mathrm{num}(\mathrm{str}(n)) = n \leq 2^{|\mathrm{str}(n)|+1} - 2.$$

Let φ be any fixed recursive numbering, let

$$\langle a, b\rangle = \mathrm{num}(1^{|\mathrm{str}(a)|}0\mathrm{str}(a)\mathrm{str}(b))$$

in concatenative notation. Let $\psi_{2\langle e,n\rangle}(x) = \varphi_e(n)$ for any x and $\psi_{2y+1} = \varphi_y$. Note that ψ is an acceptable numbering. Let $s(e, n) = 2\langle e, n\rangle$. Then if f is DNR with respect to ψ then f has the following property with respect to φ:

$$f(s(e, n)) \neq \varphi_e(n).$$

Indeed,

$$f(s(e, n)) = f(2\langle e, n\rangle) \neq \psi_{2\langle e,n\rangle}(2\langle e, n\rangle) = \varphi_e(n).$$

Moreover,

$$\begin{aligned} s(a, b) = 2\langle a, b\rangle \leq 2(2^{1+|1^{|\mathrm{str}(a)|}0\mathrm{str}(a)\mathrm{str}(b)|}) &= 8(2^{|1^{|\mathrm{str}(a)|}|}2^{|\mathrm{str}(a)|}2^{|\mathrm{str}(b)|}) \\ = 8(2^{|\mathrm{str}(a)|}2^{|\mathrm{str}(a)|}2^{|\mathrm{str}(b)|}) &\leq 8(a+1)^2(b+1). \end{aligned}$$

Consider a partition of ω into intervals I_m such that $|I_m|$ is $2 + \log(m + 1)$ rounded down, and let $h(m) = |I_m|$. If f is DNR_h^ψ then we have

$$\forall \varphi_e \, \forall n \, (f(s(e, n)) \in \{0, 1\}^{I_{s(e,n)}} \text{ and } f(s(e, n)) \neq \varphi_e(n)).$$

Given a recursive set R, there is, by the fixed-point theorem, an index e such that, for all n, $\varphi_e(n) = R \upharpoonright I_{s(e,n)}$ and $f(s(e, n)) \neq R \upharpoonright I_{s(e,n)}$. Note that for every fixed e,

$$\prod_{n=0}^{\infty}(1 - 2^{-|I_{s(e,n)}|}) \leqslant \prod_{n=e+2}^{\infty}(1 - 2^{-(2+\log(8(e+1)^2(n+1)+1))})$$

$$\leq \prod_{n=e+2}^{\infty}(1 - 2^{-(3+\log(8(e+1)^2(n+1)))}) = \prod_{n=e+2}^{\infty}(1 - 2^{-(6+2\log(e+1)+\log(n+1))}).$$

The last product in this formula is 0, as the sum

$$\sum_{n=e+2}^{\infty} 2^{-(6+2\log(e+1)+\log(n+1))} = \frac{1}{64} \cdot (e+1)^{-2} \cdot \sum_{n=e+2}^{\infty} \frac{1}{n+1}$$

diverges. Thus

$$\mu(\{B : \exists e \forall n\, [B \upharpoonright I_{s(e,n)} \neq f(s(e,n))]\}) \leqslant \sum_{e} \prod_{n=0}^{\infty} (1 - 2^{-|I_{s(e,n)}|}) = 0.$$

So if f is A-recursive then we have a $\Sigma_2^0(A)$ null set that contains all recursive sets, as desired. □

Theorem 9 (affirmative answer to Brendle, Brooke-Taylor, Ng and Nies [1, Question 4.1(7)])**.** *There exists a set A satisfying the following conditions:*

1. *A is weakly meager covering;*
2. *A does not compute any Schnorr random set;*
3. *A is of hyperimmune-free degree;*
4. *A is weakly Schnorr covering.*

Proof. Let h and ψ as in Theorem 8. By Theorem 6, there is an $f \in \mathrm{DNR}_h^{\psi}$ such that f computes no Kurtz random real. Let A be a set Turing equivalent to f.

1. By Theorem 4, A is weakly meager covering. Alternatively, one could use the fact that every weakly Kurtz covering oracle is also weakly meager covering and derive the item 1 from the proof of item 4.
2. Since each Schnorr random real is Kurtz random, A does not compute any Schnorr random real.
3. Since A does not compute any Kurtz random real, A is of hyperimmune-free degree.
4. By Theorem 8, A is weakly Kurtz covering. In particular, by Proposition 7, A is weakly Schnorr covering.

This completes the proof. □

The following proposition is well-known and will be used in various proofs below.

Proposition 10. *If A is of hyperimmune-free Turing degree and $B \leqslant_T A$ then $B \leqslant_{tt} A$.*

Franklin and Stephan [8] gave the following characterisation: A set A is *Schnorr trivial* iff for every $f \leqslant_{tt} A$ there is a recursive function g such that, for all n, $f(n) \in \{g(n,0), g(n,1), \ldots, g(n,n)\}$; this characterisation serves here as a definition.

Theorem 11 (affirmative answer to Brendle, Brooke-Taylor, Ng and Nies [1, Question 4.1(9)]). *There is a hyperimmune-free oracle A which is not* DNR *(and thus low for weak* 1*-genericity) and which is not Schnorr trivial and which does not Schnorr cover all recursive sets.*

Proof. We show that there is a set A such that the following conditions hold:

1. A is not DNR;

2. A does not have hyperimmune degree;

3. A is not Schnorr trivial;

4. A is not weakly Schnorr covering.

To this end, a partial-recursive $\{0,1\}$-valued function ψ is constructed such that every total extension A of hyperimmune-free degree satisfies the conditions that A is not DNR, not Schnorr trivial and not weakly Schnorr covering. The property that A is not Schnorr trivial is obtained by showing that there is an A-recursive function f such that $C(f(x)) > x$ for infinitely many x. (Here, C denotes the plain Kolmogorov complexity. In what follows, by Kolmogorov complexity we will always mean the plain complexity.) The property that A is not weakly Schnorr covering will be obtained by showing that there is no martingale tt-reducible to A and no recursive bound such that the martingale Schnorr succeeds on all recursive sets using this bound. Note that since A is of hyperimmune-free degree, by Proposition 10 it is sufficient to consider tt-reductions instead of Turing-reductions here.

The basic idea is to construct the partial recursive function ψ such that its domain at every stage s is the complement of the currently active intervals I_n. Here $I_0 = \{0,1\}$ and, for $n > 0$, $I_n = \{2^n+1, 2^n+2, \ldots, 2^{n+1}\}$. When ψ becomes defined on some interval I_n by setting it nonactive, ψ takes on I_n a characteristic function $\sigma \in \{0,1\}^{I_n}$ which has not been killed previously by the construction. At each stage t the following activities will be carried out:

- Select the requirement of highest priority which needs attention and is permitted to act;

- For the reserved interval I_n, find the next interval I_m which should be active and which has to be so large that one can satisfy the growth requirements of the martingale to not succeed by making the I_o with $n < o < m$ to be non-active and by later killing certain $\sigma \in \{0,1\}^{I_n}$ (see below);

- Make ψ defined on all intervals I_o with $n < o < m$;

- Update the tree T_t so that it takes the new ψ_t but only those σ killed before stage t into account: The tree T_t has those infinite branches $\tilde{A}$ which extend ψ_t and which, on any active I_n, do not take a value σ which has been killed prior to stage t;

- Kill every $\sigma \in \{0,1\}^{I_n}$ which needs to be killed according to the selected requirement and which has not been killed before (this depends on T_t);
- Make I_m to be the reserved interval for the requirement;
- For all active I_o with $o < t$ and all $\sigma \in \{0,1\}^{I_o}$, if it is found within t steps that the conditional Kolmogorov complexity of σ given o is strictly below $2^o - 1$ then kill σ (if not already done so before);
- Initialise and cancel requirements as needed for requirements $R_{k',c'}$ with $k' < t$ and $c' \leqslant k'$.

An oracle $\tilde{A}$ is valid at t iff it is an infinite branch of T_t and it is valid if it is an infinite branch of $T = \cap_t T_t$. The tree T will have infinite hyperimmune-free branches A and it will be shown that any such branch A is neither Schnorr trivial nor DNR nor weakly Schnorr covering.

Now some more details are given for the requirements. For these there is a list (M_k, f_k) of martingales M_k given by truth-table reductions to oracles and of a recursive bound functions f_k; though one cannot avoid that partial truth-table reductions and bound functions are in the list, one can nevertheless make the list in a way that one can check for each σ, ℓ and t whether $M_k(\sigma)$ is defined within t steps and whether $f_k(0), \ldots, f_k(\ell)$ are all defined within t steps. Note that only the total (M_k, f_k) are relevant and that the others will get stuck somewhere in the construction and will be ignored by all sufficient large instantiations of the requirements with true parameters. Now (M_k, f_k) succeeds on a recursive set B iff there are infinitely many n such that $M_k(B(0)B(1)\ldots B(f_k(n))) > n$; it is therefore the goal of the construction to prevent this from happening and to construct together with M_k a recursive set B (depending on k as well) such that (M_k^A, f_k) does not succeed on B. For the construction, let $use_k(x)$ denote the first time $t > x$ is found such that f_k is defined on all $y \leqslant x$ and $M_k^{\tilde{A}}(\sigma)$ is defined for all σ up to length x by querying only values below t, independently on which oracle $\tilde{A}$ is used; $use_k(x) = \infty$ if some of the above mentioned computations do not terminate. In the following, t will always be the number of the current stage and the requirements will explicitly check that the use of those members of the list which are considered to be valid is below the current stage number t on the relevant inputs.

One can without loss of generality assume that each M_k is a savings martingale so that it never goes down by more than 1 and that the functions f_k are strictly monotonically increasing; furthermore, the functions and martingales are either total or defined up to a certain point and undefined from then onwards. Franklin and Stephan [8] provide more details on such type of martingales. If $\tilde{A}$ is hyperimmune-free then this list is sufficient to deal with all relevant martingales as one can replace martingales by saving martingales and then the bound by a recursive upper bound. For each k there are exactly $k+1$ many requirements $R_{k,0}, R_{k,1}, \ldots, R_{k,k}$ for (M_k, f_k). The requirement $R_{k,c}$ is said to have true parameters iff there are exactly $c+1$ indices $e \in \{0, 1, \ldots, k\}$ with M_e, f_e being total and k is one of these. Note that when M_e, f_e are total then M_e has to be a savings martingale as described above and f_e has to be a strictly monotonically increasing recursive function. At a stage t, a requirement $R_{k,c}$ can (a) be initialised, (b) be cancelled, (c) require attention or (d) act.

Initialisation: A requirement $R_{k,c}$ can be initialised at stage t and request an interval I_q (so that q denotes from now on the index of that interval which $R_{k,c}$ took while being initialised) iff there are exactly c numbers $k_0, k_1, \ldots, k_{c-1}$ with $k_0 < k_1 < \ldots < k_{c-1} < k$ such that the following conditions hold:

- $q > k$ and $2^{|I_q|} \cdot r_k$ is an integer (for the sequence of r_k defined below);
- for each $e \leqslant k$, $use_e(\max(I_q)) < t$ iff $e \in \{k_0, k_1, \ldots, k_{c-1}, k\}$;
- for all $c' < c$, the requirement $R_{k_{c'},c'}$ is currently active and has reserved some interval I_o with $o > q$;
- I_q is active and all intervals I_o on which $R_{k,c}$ has acted in prior stages satisfy $o < q$.

Let $E_{b,t}$ contain all oracles $\tilde{A}$ such that $\tilde{A}$ is on T_t and for all D on T_t which coincide with $\tilde{A}$ below the given bound $use_k(b)$, $\tilde{A} \leqslant_{lex} D$, that is, $\tilde{A}$ is the least representative of the class of oracles which do not differ below $use_k(b)$ from $\tilde{A}$. Now let

$$N_{k,b,t}(\sigma) = \sum_{\tilde{A} \in E_{b,t}} M_k^{\tilde{A}}(\sigma)$$

for all σ up to length b. Now define B up to the maximal value x with $use_k(x) < \min(I_q)$ as taken such that $N_{k,x,t}$ does not grow and let

$$u = \max\{M_k^{\tilde{A}}(B(0)B(1)\ldots B(x)) : \tilde{A} \in E_{x,t}\}.$$

The values u, x are updated and B defined on more places when the requirement acts.

Cancellation: A requirement $R_{k,c}$ gets cancelled if there are more than c numbers $e < k$ for the initial interval I_q on which $R_{k,c}$ got initialised for the current run.

Attention: Now let $r_0, r_1, \ldots$ be a recursive sequence of negative powers of 2 which converge from above to 0 and have the property that the sum of the $r_k \cdot (k+1)$ for $k = 0, 1, 2, \ldots$ is 1/2. For a requirement $R_{k,c}$ currently having reserved an interval I_n, recall that x is the place up to which the recursive set B of the requirement has been defined and u is the maximal value $M^{\tilde{A}}(B(0)B(1)\ldots B(x))$ takes for some $\tilde{A} \in T_t$ (assuming that ψ does not get defined on intervals below I_n which will not happen in the case that $R_{k,c}$ acts). Now the requirement $R_{k,c}$ needs attention if there is an interval I_m such that

- I_n and I_m are both active and $n < m$;
- $R_{k,c}$ has currently reserved I_n;
- $use_e(\max(I_q)) > t$ for all $e \in \{0, 1, \ldots, k-1\} - \{k_0, k_1, \ldots, k_{c-1}\}$ for the initial interval I_q on which the current run of the requirement $R_{k,c}$ was initialised;
- the requirements $R_{k_e,e}$ with $e < c$ have currently reserved some interval I_o with $o > m$;
- the maximal value x' with $use_k(x') < \min(I_m)$ satisfies $x' > \max(I_n)$ and $x' > f_k((u+1) \cdot 4^{\max(I_n)}/r_k)$.

Acting: If $R_{k,c}$ receives attention, it acts as follows, where the parameters B, x, x' are as under the item "Attention".

- All I_o with $n < o < m$ will be set non-active (if not done before) and ψ_t will be defined on these intervals and T_t will be updated as outlined above;
- Let $u = \max\{M_k^{\tilde{A}}(B(0)B(1)\ldots B(x))\colon \tilde{A}$ is an infinite branch of $T_t\}$;
- $N_{k,x',t}$ will be computed and B will be extended from the domain up to x to the domain up to x' in the way that $N_{k,x',t}$ does not grow;
- For each $\sigma \in \{0,1\}^{I_n}$ let $u_\sigma = \max\{M_k^{\tilde{A}}(B(0)B(1)\ldots B(x'))\colon \tilde{A}$ is on T_t and extends $\sigma\}$ — once this is defined, one kills those $r_k \cdot 2^{|I_n|}$ strings $\sigma \in \{0,1\}^{I_n}$ for which u_σ is maximal;
- Let $u' = \max\{M_k^{\tilde{A}}(B(0)B(1)\ldots B(x'))\colon \tilde{A}$ is on T_t and $\tilde{A}$ restricted I_n has not been killed in the previous step$\}$, that is, u' is bounded by the value number $r_k \cdot 2^{|I_n|} + 1$ in a list of all the u_σ considered, in descending order;
- The new value of x is the current x' and the new interval selected for $R_{k,c}$ is I_m and the new value of u is u'.

For the verification, it first should be noted that for each I_n, at most $2^{|I_n|} - 1$ many $\sigma \in \{0,1\}^{I_n}$ get killed and therefore the amount of σ available is never exhausted. The reason is that the requirements kill at most $2^{|I_n|} \cdot \sum_{k,c} r_k = 2^{|I_n|-1}$ many σ and the Kolmogorov complexity condition at most $2^{|I_n|-1} - 1$ many σ, so at least one σ remains. Thus the ψ can on each I_n get defined when I_n is set to be non-active and the tree T_t has in each step and also in the limit infinite branches. So there is a hyperimmune-free set A on the tree T.

Second there are infinitely many I_n which remain active forever. Assume that it is shown that the interval I_n is never set to inactive. One can see that every interval gets only finitely often reserved by a requirement and therefore it happens only finitely often that a requirement acts with the interval I_n or a smaller one being the reserved interval; when this has happened for the last time, there is a larger interval I_m such that I_m is the least current active interval above I_n. From now on, it only happens that some interval I_m or beyond will be the reserved interval of a requirement which is going to act and therefore I_m will never be set inactive. So one can prove by induction that infinitely many intervals will remain active forever.

Third the resulting set is not Schnorr trivial as there are infinitely many intervals I_n from $2^n + 1$ to 2^{n+1} which remain active forever and on them, A restricted to I_n has at least the Kolmogorov complexity $2^n - 1$ conditional to n; thus the set A is not Schnorr trivial.

Fourth, let $k_0, k_1, \ldots$ be the (noneffective) sublist of all pairs (M_k, f_k) such that M_k is a total truth-table reduction giving a savings martingale and f_k is a total recursive function. Now one shows by induction over c that each requirement $R_{k_c,c}$ gets only finitely often cancelled and is eventually permanently initialised and acts infinitely often. Assume that the stage t is so large that the following conditions hold:

- the pair $(M_{k'}, f_{k'})$ with $k' \in \{0, 1, \ldots, k_c\} - \{k_0, k_1, \ldots, k_c\}$ have reached their first undefined places and let y be the maximum of these places;
- all cancellations of $R_{k_c,c}$ due to these k' have already occurred;
- the requirements $R_{k_{c'},c'}$ with $c' < c$ are all initialised and will not be cancelled after stage t and will act infinitely often;
- there is an active interval I_n such that neither I_n nor any larger interval has so far been reserved by $R_{k_c,c}$ and all requirements $R_{k_{c'},c'}$ with $c' < c$ have currently reserved some interval beyond I_n.

Then the requirement $R_{k_c,c}$ will be initialised, for example on I_n; it will not be cancelled again. Now one needs to show that it acts infinitely often; assume by way of contradiction that the requirement would remain forever on an interval I_n without acting. There is an interval I_m beyond I_n such that I_m is active forever and all the conditions of the request of attention are satisfied except the first one – this is due to selecting an I_m with sufficiently large index / position. Now, by induction hypothesis, the requirements $R_{k_{c'},c'}$ will act often enough so that they eventually reserve intervals beyond I_m and from that time onwards $R_{k_c,c}$ will require attention and therefore eventually act.

Fifth: If M_k, f_k are total and c is chosen such that (k, c) are true parameters then the set B constructed by requirement $R_{k,c}$ in its infinite run is not covered by the martingale M_k with bound f_k in the Schnorr sense. There is a case distinction between the case where, in a run, the requirement $R_{k,c}$ acts for the first and for a subsequent time.

Assume now that $R_{k,c}$ acts for the first time. Let x be the value up to which B has been defined in the initialisation and let u be the corresponding maximum value taken by some martingale up to x on B which is valid at the time of initialisation. Note that ψ_t will be defined in stage t on all values strictly between $\max(I_n)$ and $\min(I_m)$. Then $N_{k,x',t}$ is the sum of at most $2^{\max(I_n)}$ martingales and B is chosen on the values from $x + 1$ up to x' such that $N_{k,x',t}$ does not increase; hence the value $N_{k,x',t}(B(0)B(1)\ldots B(x'))$ is bounded by

$2^{\max(I_n)} \cdot u$ and therefore each outgoing martingale $M_k^{\tilde{A}}$ satisfies

$$M_k^{\tilde{A}}(B(0)B(1)\ldots B(x')) \leqslant 2^{\max(I_n)} \cdot u$$

while at the same time x' satisfies

$$f_k((u+1) \cdot 4^{|I_n|}/r_k) < x'$$

and thus, for the new bound u',

$$f_k((u'+1) \cdot 2^{I_n}/r_k) < x'$$

which can be used as an incoming bound for subsequent actions of the requirement $R_{k,c}$.

If now $R_{k,c}$ acts for a subsequent time in the run of a requirement, then one verifies besides the above assurance on the outgoing bound – the proof goes through unchanged – also that the martingale cannot succeed in the Schnorr sense between x and x' where x' is the new point up to which B gets defined during the acting.

Now let I_o be the interval on which it acted before it acts on I_n and let I_m be the interval where it scheduled to act next (though the interval might actually be larger), that is I_n and I_m are the parameters used during the current acting of the requirement. Note that $o < n < m$. Furthermore, when the requirement acts on I_n then I_n is the first active interval after I_o and therefore ψ is defined before stage t between $\max(I_o)$ and $\min(I_n)$ and it will become defined between $\max(I_n)$ and $\min(I_m)$ during the stage t or already before stage t. Furthermore the sum of the martingales $N_{k,x',t}$ (with the parameters defined as in the proof) only need to take into account the oracles which coincide with ψ_t as only those can be identical with the A as A is on T. Let x be the bound to which B is defined before I_n acts and x' be the bound after I_n acts; furthermore, u and u' are defined accordingly and t is the time when $R_{k,c}$ acts on I_n. By induction hypothesis, $f_k((u+1) \cdot 2^{\max(I_o)}/r_k) < x$. Therefore one has only to show that

$$M_k^{\tilde{A}}(B(0)B(1)\ldots B(x')) < (u+1) \cdot 2^{\max(I_o)}/r_k$$

in order to satisfy the constraint on non-success for all $\tilde{A}$ on T_t which do not get $\tilde{A}$ restricted to I_n killed in stage t. This can be seen as $N_{k,t,x'}$ does not go up on B from x to x' due to the choice of B and furthermore there are at most $2^{\max(I_o)} \cdot 2^{|I_n|}$ many initial segments $\tau = \tilde{A}(0)\tilde{A}(1)\ldots\tilde{A}(\min(I_m)-1)$ which have to be taken into account to compute the value of the sum $N_{t,x',k}(B(0)B(1)\ldots B(x'))$; among these values, the largest $r_k \cdot 2^{|I_n|}$ many terms in the sum will be removed from it due to the killing of the corresponding σ; it follows that the maximum u' of the remaining terms satisfies

$$u' \leqslant u \cdot \frac{2^{\max(I_o)} \cdot 2^{|I_n|}}{r_k \cdot 2^{|I_n|}} \leqslant u \cdot 2^{\max(I_o)}/r_k$$

which satisfies the required bound. This calculation is based on the fact that if there are up to a values whose sum bounded by $u \cdot a$ and one kills the largest

b of these values then the remaining values are each are bounded by $u \cdot a/b$, as otherwise the b killed values would each be strictly above $u \cdot a/b$ and have a sum strictly above $b \cdot u \cdot a/b = u \cdot a$ what is impossible by assumption on $u \cdot a$ being the sum of all of the values. Now the $2^{\max(I_o)} \cdot 2^{|I_n|}$ in the numerator is an upper bound on the overall number of terms to be considered and $2^{|I_n|} \cdot r_k$ is a lower bound on the number of largest terms to be removed from the sum which follows from the overall number of new strings killed in this iteration. Thus none of the surviving oracles $\tilde{A}$ satisfies

$$M_k^{\tilde{A}}(B(0)B(1)\ldots B(x')) > u \cdot 2^{\max(I_o)}/r_k$$

and so the martingale is below the value $u \cdot 2^{\max(I_o)}/r_k + 1$ on all prefixes of $B(0)B(1)\ldots B(x')$. Thus the growth bound is maintained between x and x'. In particular it follows that the growth bound on M_k^A is maintained at every acting of the requirement except for the first after the initialisation. Therefore (M_k^A, f_k) does not Schnorr succeed on the recursive set B.

Sixth the set A is not DNR. To see this, recall that for being DNR and hyperimmune-free there needs to be a recursive function g such that A up to $g(n)$ has at least Kolmogorov complexity n for every n [11]; without loss of generality g can be taken to be strictly monotonically increasing. There is a k such that $f_k = g$ and $M_k^A(\sigma)$ a martingale which always bets 0. There is a corresponding requirement $R_{k,c}$ which acts infinitely often. When acting with I_n being a reserved interval, the requirement ensures that there is another interval I_m such that $f_k(2^{\max(I_n)}/r_k) < \min(I_m)$ and ψ is defined between $\max(I_n)$ and $\min(I_m)$. It follows that the Kolmogorov complexity of A up to $f_k(2^{\max(I_n)})$ is at most $\max(I_n) + n + O(1)$ for this I_n and infinitely many other I_n, thus the constraint is violated and A is not DNR. This completes the proof. $\square$

Remark 12. The reader may object that the original question in [1] asked for a set that was *not low for Schnorr tests* rather than *not Schnorr trivial*. However, we can recall the following facts:

- Kjos-Hanssen, Nies and Stephan [12] showed that if A is low for Schnorr tests then A is low for Schnorr randomness;
- Franklin [7] showed that if A is low for Schnorr randomness then A is Schnorr trivial.

3 Infinitely Often Subuniformity and Covering

Let $\langle .,. \rangle$ denote a standard recursive bijection from $\omega \times \omega$ to ω. For a function $P : \omega \to \omega$ define

$$P_n(m) = P(\langle n, m \rangle)$$

and say that P *parametrises* the class of functions $\{P_n : n \in \omega\}$. We identify sets of natural numbers with their characteristic functions. A class $\mathcal{A}$ is *(recursively) uniform* if there is a recursive function P such that $\mathcal{A} = \{P_n : n \in \omega\}$, and *(recursively) subuniform* if $\mathcal{A} \subseteq \{P_n : n \in \omega\}$. These notions relativise to any oracle A to yield the notions of *A-uniform* and *A-subuniform*.

It is an elementary fact of recursion theory that the recursive sets are not uniformly recursive. The following theorem, as cited in Soare's book [21, page 255], quantifies exactly how difficult it is to do this:

Theorem 13 (Jockusch)**.** *The following conditions are equivalent:*

(i) *A is high, that is, $A' \geqslant_T \emptyset''$,*

(ii) *the recursive functions are A-uniform,*

(iii) *the recursive functions are A-subuniform,*

(iv) *the recursive sets are A-uniform.*

If A has r.e. degree then (i)–(iv) *are each equivalent to:*

(v) *the recursive sets are A-subuniform.*

In the following we study infinitely often parametrisations and the relation to computing covers for the recursive sets.

3.1 Infinitely Often Subuniformity

The next definition generalises from "Schnorr covering" to "covering" which just says that a martingale succeeds on all sets of the class (without having a bound on the time until it has to succeed infinitely often).

Definition 14. We say that a set X *covers* a class $\mathcal{A}$ if there is an X-recursive martingale M such that $\mathcal{A} \subseteq S[M]$.

Note that for X recursive this is just the definition of recursive measure zero.

Definition 15. A class $\mathcal{A} \subseteq 2^\omega$ is called *infinitely often subuniform* (i.o. subuniform for short) if there is a recursive function $P \in \{0,1,2\}^\omega$ such that

$$\forall A \in \mathcal{A}\, \exists n \left[\exists^\infty x\big(P_n(x) \neq 2\big) \wedge \forall x\big(P_n(x) \neq 2 \rightarrow P_n(x) = A(x)\big)\right].$$

That is, for every $A \in \mathcal{A}$ there is a row of P that computes infinitely many elements of A without making mistakes. Again, we can relativise this definition to an arbitrary set X: A class $\mathcal{A}$ is i.o. X-subuniform if P as above is X-recursive.

Let REC denote the class of recursive sets. Recall that A is a PA-complete set if A can compute a total extension of every $\{0,1\}$-valued partial recursive function. Note that if a set A is PA-complete then REC is A-subuniform (see Proposition 16 below).

For every recursive set A there is a recursive set $\hat{A}$ such that A can be reconstructed from any infinite subset of $\hat{A}$. Namely, let $\hat{A}(x) = 1$ precisely when x codes an initial segment of A. So it might seem that any i.o. sub-parametrisation of REC can be converted into a sub-parametrisation in which every recursive set is completely represented. However, we cannot do this uniformly (since we cannot get rid of the rows that have $P_n(x) = 2$ a.e.) and indeed the implication does not hold.

Proposition 16. *We have the following picture of implications:*

$$\begin{array}{ccccc} & & A \text{ is high} & \Rightarrow & A \text{ has hyperimmune degree} \\ & & \Downarrow & & \Downarrow \\ A \text{ is PA-complete} & \Rightarrow & \mathrm{REC} \text{ is } A\text{-subuniform} & \Rightarrow & \mathrm{REC} \text{ is i.o. } A\text{-subuniform} \end{array}$$

No other implications hold than the ones indicated.

Proof. The proposition follows from the following observations.

If A is PA-complete then it can in particular compute a total extension of the universal $\{0,1\}$-valued partial-recursive function, hence compute a list of total functions in which every $\{0,1\}$-valued recursive function appears.

If A is of hyperimmune degree there is an A-recursive function that is not dominated by any recursive function. This function can be used to compute infinitely many points from every recursive set, in a uniform way. More precisely, let $f \leqslant_T A$ be a function that is not recursively dominated. If φ_e is total then also $\Phi(x) = \mu s.\varphi_{e,s}(x)\downarrow$ is total, hence $f(x) \geqslant \Phi(x)$ and $\varphi_{e,f(x)}(x)\downarrow$ infinitely often. For these x, let $P_e(x) = \varphi_e(x)$; for the other x, let $P_e(x) = 2$. Then $P \leqslant_T A$ is a parametrisation and if φ_e is total then $P_e(x) = \varphi_e(x)$ infinitely often.

To see that REC A-subuniform does not imply that A is PA-complete, first note that PA-complete sets cannot have incomplete r.e. degree by a result of Scott and Tennenbaum [17, p513]. Second, by Theorem 13, if A is high then REC is A-uniform. So the non-implication follows from the existence of a high incomplete r.e. set (Sacks, see [18, p650]).

To see that A having hyperimmune degree does not imply that REC is A-subuniform, note that again by Theorem 13 we have for A r.e. that REC is A-subuniform implies that A is high. Now let A be r.e. nonrecursive (so that in particular A has hyperimmune degree [17, p495]) and non-high. Then REC is not A-subuniform. In particular, we see from this non-implication that i.o. subuniformity of REC does not imply subuniformity.

Finally, it is well-known that A PA-complete does not imply that A has hyperimmune degree (and hence the weaker notions in the diagram also do not imply it): The PA-complete sets form a Π^0_1 class, hence, since the sets of hyperimmune-free degree form a basis for Π^0_1 classes [17, p509], there is a PA-complete set of hyperimmune-free degree. □

Proposition 17. *Every i.o. subuniform class has recursive measure zero. This relativises to: If $\mathcal{A}$ is i.o. X-subuniform then X covers $\mathcal{A}$.*

Proof. The ability to compute infinitely many bits from a set clearly suffices to define a martingale succeeding on it. The uniformity is just what is needed to make the usual sum argument work. □

Proposition 18. *There exists a class of recursive sets that has recursive measure zero and that is not i.o. subuniform.*

Proof. The class of all recursive sets A satisfying $\forall x\,[A(2x) = A(2x+1)]$ has recursive measure 0 but is not i.o. subuniform: If P would witness this class to be i.o. subuniform then Q defined as $Q_i(x) = \min\{P_i(2x), P_i(2x+1)\}$ would witness REC to be i.o. subuniform, a contradiction. □

Above the recursive sets, the 1-generic sets are a natural example of such a class that has measure zero but that is not i.o. subuniform: It is easy to see that the 1-generic sets have recursive measure zero because for every such set A there are infinitely many n such that $A \cap \{n, n+1, \dots, 2n\} = \emptyset$. On the other hand, a variation of the construction in the proof of Proposition 18 shows that the 1-generic sets are not i.o. X-subuniform for any X:

Proposition 19. *The 1-generic sets are not i.o. X-subuniform for any set X.*

Proof. Let $P \subseteq \{0,1,2\}^\omega$ be an X-recursive parametrisation and let A be 1-generic relative to X (so that A is in particular 1-generic). Then for every n, if $P_n(x) \neq 2$ for infinitely many x then

$$\left\{ \sigma \in 2^{<\omega} : \exists x\,[P_n(x) \neq 2 \wedge P_n(x) \neq \sigma(x)] \right\}$$

is X-recursive and dense, hence A meets this set of conditions and consequently P does not i.o. parametrise A. □

Now both the example from Proposition 18 and the 1-generic sets are counterexamples to the implication "measure 0 $\Rightarrow$ i.o. subuniform" because of the *set structure* of the elements in the class. One might think that for classes closed downwards under Turing reducibility (that is, for classes defined by information content rather than set structure) the situation could be different. For example, one might conjecture that for $\mathcal{A}$ closed downwards under Turing reducibility, the implication "X covers $\mathcal{A} \Rightarrow \mathcal{A}$ i.o. X-subuniform" would hold. Note that for X recursive this is not interesting, since any nonempty class closed downwards under Turing reducibility contains REC and REC does not have recursive measure zero. However, this conjecture is also not true: Consider the class

$$\mathcal{A} = \{A : A \leqslant_T G \text{ for some 1-generic } G\}.$$

Clearly $\mathcal{A}$ is closed downwards under Turing reducibility and it follows from proofs by Kurtz [13] and by Demuth and Kučera [4] (a proof is also given by Terwijn [22]), that $\mathcal{A}$ is a Martin-Löf null set and that in particular the halting problem K covers $\mathcal{A}$. However, by Proposition 19 the 1-generic sets are not i.o. K-subuniform so that in particular $\mathcal{A}$ is not i.o. K-subuniform.

3.2 A Nonrecursive Set that does not Cover REC.

It follows from Proposition 16 and Proposition 17 that if A is of hyperimmune degree then A covers REC. In particular every nonrecursive set comparable with K covers REC. We see that if A cannot cover REC then A must have hyperimmune-free degree. We now show that there are indeed nonrecursive sets that do not cover REC. Indeed, the following result establishes that there are natural examples of such sets; this result can be seen as a generalisation of the result of Calude and Nies [3] that Chaitin's Ω is wtt-complete and tt-incomplete; see Nies' book [16, Theorem 4.3.9] for more information.

Theorem 20. *If A is Martin-Löf random then there is no martingale $M \leqslant_{tt} A$ which covers* REC. *In particular if A is Martin-Löf random and of hyperimmune-free Turing degree then it does not cover* REC.

Proof. Let A be Martin-Löf random and M^A be truth-table reducible to A by a truth-table reduction which produces on every oracle a savings martingale, that is, a martingale which never goes down by more than 1. Without loss of generality, the martingale starts on the empty string with 1, takes rational values and is never less than or equal to 0. Note that because of the truth-table property, one can easily define the martingale N given by

$$N(\sigma) = \int_{E \subseteq \omega} M^E(\sigma)\, dE,$$

where the integration "dE" weights all oracles with the uniform Lebesgue measure. As one can replace the E by the strings up to $use(|\sigma|)$ using the recursive use-function use of the truth-table reduction, one has that

$$N(\sigma) = \sum_{\tau \in \{0,1\}^{use(|\sigma|)}} 2^{-|\tau|} M^\tau(\sigma)$$

and N is clearly a recursive martingale; also the values of N are rational numbers. Let B be a recursive set which is adversary to N, that is, B is defined inductively such that

$$\forall n\, [N(B{\upharpoonright}(n+1)) \leqslant N(B{\upharpoonright}n)].$$

Define the uniformly r.e. classes S_n by

$$S_n = \{E : M^E \text{ reaches on } B \text{ a value beyond } 2^n + 1\}.$$

By the savings property, once M^E has gone beyond $2^n + 1$ on B, M^E will stay above 2^n afterwards. It follows that the measure of these E can be at most 2^{-n}. So $\mu(S_n) \leqslant 2^{-n}$ for all n and therefore the S_n form a Martin-Löf test. Since A is Martin-Löf random, there exists n such that $A \notin S_n$, and hence M^A does not succeed on B. □

The anonymous referee pointed out to the authors that Theorem 20 has a variant which is true when A is Kurtz random. The precise statement is the following:

Proposition 21. **(a)** *If A is Kurtz random then there is no martingale $M \leqslant_{tt} A$ and no recursive bound function f which Kurtz cover* REC*, that is, which satisfy that for all $B \in$ REC and almost all n, $M(B(0)B(1)\dots B(f(n))) \geqslant n$.*

(b) *If A is Schnorr random then there is no martingale $M \leqslant_{tt} A$ and no recursive bound function f which Schnorr cover* REC*, that is, which satisfy that for all $B \in$ REC and for infinitely many n, $M(B(0)B(1)\dots B(f(n))) \geqslant n$.*

Note that a weakly 1-generic set A is Kurtz random and coincides on arbitrarily long parts with any given recursive set B, so the martingale which bets half of the capital on the next digit of A and B to be the same will Schnorr cover all recursive sets B. Therefore one has to use "Kurtz cover" for part (a). The observation of the referee allows then to conclude that the truth-table degrees of weakly 1-generic and Schnorr random sets can never be the same, as the first ones Schnorr cover REC and the second ones don't.

We note that the set $\{A \in \{0,1\}^\omega : A \text{ covers REC}\}$ has measure 1. This follows from Proposition 16 and the fact that the hyperimmune sets have measure 1 (a well-known result of Martin, see [6, Theorem 8.21.1]).

We note that apart from the hyperimmune degrees, there are other degrees that cover REC.

Proposition 22. *There are sets of hyperimmune-free degree that cover the class* REC*.*

Proof. As in Proposition 16, take a PA-complete set A of hyperimmune-free degree. Then the recursive sets are A-subuniform, so by Proposition 17 A covers REC. □

3.3 Computing Covers versus Uniform Computation

We have seen above that in general the implication "X covers $\mathcal{A} \Rightarrow \mathcal{A}$ i.o. X-subuniform" does not hold, even if $\mathcal{A}$ is closed downwards under Turing reducibility. A particular case of interest is whether there are sets that can cover REC but relative to which REC is not i.o. subuniform.

Theorem 23. *There exists a set A that Schnorr covers* REC *but relative to which* REC *is not i.o. A-subuniform.*

Proof. We construct the set A by choosing a total extension of hyperimmune-free Turing degree of a partial-recursive $\{0,1\}$-valued function ψ built by a finite injury construction. In the following, we will consider parametrisations computable by A. Because A is of hyperimmune-free degree, for every Turing reduction to A there is an equivalent truth-table reduction to A by Proposition 10, so it will be sufficient to only consider the latter. We will consider tt-reductions Φ^E that compute i.o. parametrisations relative to an oracle E, and we denote the i-th component of such a parametrisation by Φ_i^E.

Let $I_0 = \{0,1,2\}$ and, for $n > 0$, $I_n = \{3^n, 3^n+1, \dots, 3^{n+1}-1\}$. Now if A coincides with a set B on infinitely many intervals I_n then the martingale which always puts half of its money onto the next bit according to the value of A succeeds on B, indeed, it even Schnorr succeeds on B. The reason is that if I_n is such an interval of coincidence, then at least $3^{n+1} - 3^n$ of the bets are correct and the capital is at least $3^{3^{n+1}-3^n}/2^{3^{n+1}} = (9/8)^{3^n}$, as it multiplies with $3/2$ at a correct bet and halves at an incorrect bet. Thus the overall goal of the construction is to build a partial recursive function ψ with the following properties:

- ψ coincides with every recursive set on infinitely many I_n, and therefore every total extension E of ψ Schnorr covers REC;
- For every truth-table reduction Φ there is a recursive set B such that $\{B\}$ is not i.o. subuniform for any total extension E of ψ via Φ:

$$(*) \qquad \forall i\,(\forall^\infty x\,[\Phi_i^E(x) = 2] \vee \exists x\,[\Phi_i^E(x) = 1 - B(x)]).$$

To simplify the construction, we define a list of admissible truth-table reductions, which are all truth-table reductions Φ that satisfy one of the following two conditions:

(1) Φ is total for all oracles and computes a sequence $\Phi_0, \Phi_1, \ldots$ of $\{0, 1, 2\}$-valued functions such that for all oracles E, Φ_{2i}^E is the characteristic function of the i-th finite set.

(2) Φ_i^E is partial for all i and all E, and the set $\{(i, x) : \Phi_i^E(x)$ is defined for some $E\}$ is finite.

It is easy to see that there is an effective list of all admissible truth-table reductions. Condition (1) includes that all Φ_i^E for even i follow finite sets; this is needed in order to avoid that the construction of the set B gets stuck; it is easy to obtain this condition by considering a join of a given truth-table reduction with a default one computing all finite sets. The inclusion of the partial reductions is there in order to account for the fact that there is no recursive enumeration of all total recursive functions and thus also no recursive enumeration of all total truth-table reductions. So Condition (2) is needed to make the enumeration effective.

There will be actions with different priority; whenever several actions apply, the one with the highest priority represented by the lowest natural number will be taken. Here ψ_s at stage s is for each interval either defined on the whole interval or undefined on the whole interval, and it is defined only on finitely many intervals; furthermore, J_n refers to the n-th interval where ψ_s is undefined and c_n refers to the number of arguments where ψ_s is undefined below $\min(J_n)$. The following actions can be taken at stage s with priority $n = \max\{i, j\}$, for the parameters i, j given below:

1. The action *requires attention* if $\varphi_{i,s}(x)$ is defined for all $x \leqslant \max(J_n)$ and if there are exactly j intervals I_k with $\max(I_k) < \min(J_n)$ and $\psi_s(y)\downarrow = \varphi_{i,s}(y)\downarrow$ for all $y \in I_k$. In this case the action requires attention with priority n. If the action receives attention then we let $\psi_{s+1}(x) = \varphi_{i,s}(x)$ for all $x \in J_n$; thus each J_m with $m \geqslant n$ will move to J_{m+1}.

2. Let Φ be the j-th admissible truth-table reduction and let x be the least value where the set B defined alongside Φ (to satisfy $(*)$ above) has not yet been defined by stage s, and let E be a total extension of ψ_s. The action requires attention if the following three conditions hold:

   - $\Phi_k^F(y)$ is defined for all oracles F, all $y \leqslant x$ and all $k \leqslant n$ by stage s with use s,
   - $\forall y < x\, [\Phi_i^E(y) \in \{B(y), 2\}]$,
   - $\Phi_i^E(x) \in \{0, 1\}$.

   If it receives attention then one defines $B(x) = 1 - \Phi_i^E(x)$ and for all $m \geqslant n$ with $\min(J_m) \leqslant s$ and all $y \in J_m$ one defines $\psi_{s+1}(y) = E(y)$ and therefore all the intervals J_m with $m \geqslant n$ are moved beyond s.

At stage s the algorithm chooses an action with highest priority (= least numerical value of the priority number) that can be taken (if any), and the algorithm does not change anything if there is no action which can be taken; in the case that several actions can be taken with the same highest priority, it uses some default ordering (length-lexicographical ordering of some coding of the actions) in order to decide which one to do. As mentioned at the beginning, we let A be any extension of the so constructed function ψ of hyperimmune-free Turing degree. Such A exists by the standard construction of a hyperimmune-free degree by Miller and Martin, see [17].

The first part of the verification consists of inductively proving the following for each n:

- The number $c_n = \sum_{m<n} |J_m|$ increases only finitely often, and after some stage s, no action of priority $m < n$ is taken and none of the intervals J_m with $m < n$ moves again;
- After this stage s, the number of times that an action of priority n will be taken is at most $(2^{c_n} + 1) \cdot (2n + 1)$, and the interval J_n will only be moved when an action of priority n acts, that is, it will also be moved only finitely often.

Note that the first item is the induction hypothesis and the proof of the second is the inductive step; for $n = 0$ the first hypothesis is void and therefore satisfied. When an action of priority n is taken, only the values of ψ in some intervals J_m with $m \geqslant n$ will be filled, and therefore only the intervals J_n and beyond will be moved.

First consider actions of priority n which are of type 1, that is, for which J_n gets defined according to some φ_i. Here $n = \max\{i, j\}$ where j is the number of intervals I_k below J_n where ψ and φ_i are both defined by stage i and equal. Whenever the action is taken and J_n is moved afterwards, the number j increases by 1; hence it happens at most $2n+1$ times that ψ is defined on J_n to be equal to some φ_i by an action of type 1: For each i there is one action in the case that $i < n$ and $j = n$, and $n + 1$ actions when $i = n$ and $j = 0, 1, \ldots, n$.

Now consider actions of priority n which are of type 2. For each pair (i,j) with $\max\{i,j\} = n$ and each E extending ψ, there is at most one action where the j-th admissible truth-table reduction Φ applies and $B(x)$ is set to be different from the value $\Phi_i^E(x)$ because all $y < x$ satisfy $\Phi_i^E(y) \in \{B(y), 2\}$; furthermore, if this action is made subsequently for two different sets E and F with the same (i,j) then $E(z) \neq F(z)$ for some $z \in \cup_{m<n} J_m$ (this set does not change after stage s) and therefore there are for each (i,j) at most 2^{c_n} such actions, giving the overall upper bound of $2^{c_n} \cdot (2n+1)$ actions of type 2 with priority n after stage s.

The sum of the two calculated upper bounds gives the overall upper bound $(2^{c_n}+1) \cdot (2n+1)$ of actions of priority n carried out after stage s. Thus the inductive step is completed.

It is clear that actions of the first type are carried out eventually for all n in the case that φ_i is total and $\{0,1\}$-valued. Thus each recursive set coincides with each total extension of ψ on infinitely many intervals; in particular A does so and therefore A is weakly Schnorr covering, as explained at the beginning of the proof.

Now consider the j-th admissible truth-table reduction and assume that it is total. We observe that the set B gets defined for every x, as for each x there exists an i such that Φ_i is $\{0,1\}$-valued and coincides with B below x, hence $B(x)$ will be defined and either diagonalise Φ_i^A at x or diagonalise some other set with some other oracle. Choose any i and let s be so large that no action of priority $\max\{i,j\}$ or less will take place at or after stage s. Then there is no $x > s$ such that the following two conditions are satisfied at the state t where $B(x)$ gets defined:

- $\Phi_i^A(x) \in \{0,1\}$ and
- $\forall y < x\, [\Phi_i^A(y) \in \{B(y), 2\}]$.

The reason is that if these two conditions would be satisfied then an action of priority $\max\{i,j\}$ would qualify and enforce that some action of this or higher priority has to be carried out; by assumption on s this does however not happen. Therefore either Φ_i^A is inconsistent with B and there is an x with $\Phi_i^A(x) = 1 - B(x)$, or all $x > s$ satisfy $\Phi_i^A(x) = 2$. We conclude that the j-th admissible reduction Φ does not witness that REC is i.o. A-subuniform and, as j was arbitrary, REC is not i.o. A-subuniform. □

Theorem 23 shows in particular that there are sets A covering REC for which REC is not i.o. A-subuniform. Thus we see that there are sets that cover REC "truly probabilistically".

Theorem 23 also has a counterpart. Recall that Terwijn and Zambella [23] showed (reformulating using the results of Franklin [7]) that no Schnorr trivial A of hyperimmune-free Turing degree is weakly Schnorr covering; actually they showed that every class of sets Schnorr covered by an A-recursive martingale M with bound f is already Schnorr covered by a recursive N with recursive bound g, and the just mentioned observation follows from the fact that there is no recursive martingale covering all recursive sets.

The next result completes the picture from Theorem 23 that the two notions "REC is i.o. A-subuniform" and "A is weakly Schnorr covering" (which means that A Schnorr covers REC) are incomparable. Note that they are both implied by PA-complete and by hyperimmune and they both imply "A covers REC".

Theorem 24. *There exists a Schnorr-trivial and hyperimmune-free set A such that* REC *is i.o. A-subuniform but A does not Schnorr cover* REC.

Proof. Let c_Ω be the modulus of convergence of Chaitin's Ω. The function c_Ω is approximable from below and dominates every recursive function. We construct a recursive function f and a partial recursive $\{0,1\}$-valued function ψ such that the following conditions are met:

- For each e there is at most one n with $f(n) = e$ and $\psi(f(n))$ being undefined;
- For each total $\{0,1\}$-valued φ_e there are infinitely many n with $f(n) = e$;
- For all n, if $c_\Omega(f(n)) \geqslant n$ then $\psi(n)$ is defined else $\psi(n) \simeq \text{Parity}(\varphi_{f(n)}(n))$.

The oracle A will then be fixed as a hyperimmune-free total extension of ψ.

The recursive function f can be defined inductively with monitoring ψ on the places below n where $f(n)$ is the coordinate e of the least pair (d, e) such that there are exactly d many $m < n$ with $f(m) = e$ and $\psi(m)$ being defined for each of these m within n computation steps. There are the following two cases:

- There are infinitely many n with $f(n) = e$. Then $\psi(n)$ is defined for all these n and $\psi(n) = \text{Parity}(\varphi_e(n))$ for almost all of these n.
- There are only finitely many n with $f(n) = e$. Then $\psi(n)$ is undefined only on the largest of these n and this n also satisfies that $\varphi_e(n)$ is undefined.

Furthermore, we define $\psi(n)$ by the first of the following two searches that halt:

- If $\varphi_{f(n)}(n)$ converges then one tries to define that $\psi(n)$ is $\text{Parity}(\varphi_{f(n)}(n))$;
- If $c_\Omega(f(n)) \geqslant n$ then one tries to define that $\psi(n) = 0$.

Thus if φ_e is $\{0,1\}$-valued and total then the first case applies and $\varphi_e(n) = \psi(n)$ for almost all n where $f(n) = e$.

One now makes a family P_d consisting of all finite variants of functions Q_e which defined which are defined as follows: If $f(n) = e$ then $Q_e(n) = A(n)$ else $Q_e(n) = 2$. Note that the Q_e are uniformly recursive in A and so are the P_d. Furthermore, as for each total and $\{0,1\}$-valued φ_e the function Q_e is correct on almost all of its infinitely many predictions, one finite variant P_d of Q_e will coincide with φ_e on all of its predictions. Thus REC is i.o. A-subuniform.

The function ψ has below $c_\Omega(e)$ only undefined places at n with $f(n) < e$ and for each possible value of f below e at most one undefined place, hence the domain of ψ is dense simple (see [18] for the definition). By a result of Franklin and Stephan [8], the total extension A of ψ is Schnorr trivial. As A has also hyperimmune-free degree, A is not weakly Schnorr covering, by the results of Terwijn and Zambella [23] discussed above. $\square$

This construction has a relation to [1, Question 4.1(8)] which could be stated as follows:

> Is there a DNR and hyperimmune-free set which neither computes a Schnorr random nor is weakly Schnorr covering?

Note that the original question of the authors asked for weakly meager covering in place of DNR; however, weakly meager covering together with not weakly Schnorr covering implies both DNR and hyperimmune-free while, for the other way round, DNR implies weakly meager covering. Thus the formulation given here is equivalent to the original question.

So let f, ψ be as in the proof of Theorem 24, and let ϑ be the following numbering: If $n \neq m$ then $\vartheta_n(m) = \varphi_{f(n)}(m)$ else $\vartheta_n(n)$ is obtained by monitoring the definition of ψ and doing the following:

- If $\psi(n)$ gets defined by following Parity($\varphi_{f(n)}(n)$) then $\vartheta_n(n) = \varphi_{f(n)}(n)$;
- If $\psi(n)$ gets defined by taking 0 due to $c_\Omega(f(n)) \geqslant n$ then $\vartheta_n(n) = 0$;
- If $\psi(n)$ does not get defined then $\vartheta_n(n)$ remains undefined.

Now consider the K-recursive function h given as follows: $h(e)$ is the first n such that $f(n) = e$ and $\vartheta_{f(n)}(n) \simeq \varphi_e(n)$ — the halting problem K allows us to check this. The construction gives that such an index is always found and therefore $\vartheta_{h(e)} = \varphi_e$. A numbering with such a K-recursive translation function is called a K-acceptable numbering. Furthermore, the mapping $n \mapsto 1 - A(n)$ witnesses that A is DNR^ϑ. Thus A satisfies the conditions from [1, Question 4.1(8)] with DNR^ϑ in place of DNR; this does not answer the original question, as DNR^ϑ is weaker than DNR.

Corollary 25. *For some K-acceptable numbering ϑ, there is a DNR^ϑ, Schnorr trivial and hyperimmune-free oracle A; such an A neither computes a Schnorr random nor is weakly Schnorr covering.*

We can now extend the picture of Proposition 16 to the following.

Theorem 26. *We have the following picture of implications:*

$$
\begin{tikzcd}
 & \textit{A is high} \arrow[r, Rightarrow] \arrow[d, Rightarrow] & \textit{A has hyperimmune degree} \arrow[d, Rightarrow] \\
\textit{A is PA-complete} \arrow[r, Rightarrow] & \text{REC } \textit{is A-subuniform} \arrow[r, Rightarrow] \arrow[d, Rightarrow] & \text{REC } \textit{is i.o. A-subuniform} \arrow[d, Rightarrow] \\
\textit{A has hyperimmune degree} \arrow[r, Rightarrow] & \textit{A Schnorr covers } \text{REC} \arrow[r, Rightarrow] & \textit{A covers } \text{REC} \arrow[d, Rightarrow] \\
 & & \textit{A is nonrecursive}
\end{tikzcd}
$$

No other implications hold besides the ones indicated; note that for having a clean graphical presentation, the notion "A has hyperimmune degree" has two entries.

Proof. The upper part of the diagram was discussed in Proposition 16. That REC i.o. A-subuniform implies that A covers REC is immediate from Proposition 17. That REC i.o. A-subuniform does not imply that A Schnorr covers REC was proven in Theorem 23. That the converse also does not hold was proven in Theorem 24.

Since by Proposition 16, A is PA-complete does not imply that A has hyperimmune degree; the same is true for all notions implied by A being PA-complete, that is, for REC being A-subuniform, for A Schnorr covering REC, for REC being i.o. A-subuniform, for A covering REC and for A being nonrecursive. Rupprecht [19, 20] proved that sets of weakly 1-generic degree – which are the same as sets of hyperimmune degree – are Schnorr covering REC.

That A nonrecursive does not imply that A covers REC follows from Theorem 20. □

The following interesting question is still open.

Question 27. Are there sets A such that A covers REC, but not the class RE of all recursively enumerable sets?

Note that this really asks for the class of all recursively enumerable sets and not the class of all left-r.e. sets; if A is low for Martin-Löf randomness and nonrecursive then A covers REC but fails to cover the left-r.e. sets, as Ω is Martin-Löf random relative to A.

Hirschfeldt and Terwijn [9] proved that the low sets do not have Δ_2^0-measure zero in Δ_2^0, that is, there does not exist a K-recursive martingale that succeeds on all the low sets, where K is the halting-problem.

The reason is that given such an martingale M^K, one can consider the variant O^K which behaves on the bits with index $2k$ like M^K on the bits with index k and which ignores the bits with indices $2k+1$; furthermore, let N^K be a martingale which covers on all sets which are not Martin-Löf random. The sum of N^K and O^K gives a K-recursive martingale which covers all sets covered by O^K or N^K. However, some K-recursive set $A \oplus B$ withstands this sum martingale. Thus $A \oplus B$ is Martin-Löf random. By van Lambalgen's Theorem, the half A of $A \oplus B$ is low and Martin-Löf random; as A consists in $A \oplus B$ of the bits with index $2k$, the construction gives that M^K does not cover A.

In particular, the low Martin-Löf random sets are not i.o. K-subuniform. Despite the non-uniformity of the low sets, Downey, Hirschfeldt, Lempp and Solomon [5] succeeded in constructing a set in Δ_2^0 that is bi-immune for the low sets.

Acknowledgements. The authors would like to thank George Barmpalias and Michiel van Lambalgen for discussions about Section 3 and André Nies and Benoit Monin for correspondence and thorough checking of Theorem 11.

References

[1] Jörg Brendle, Andrew Brooke-Taylor, Keng Meng Ng, and André Nies. An analogy between cardinal characteristics and highness properties of oracles. In *Proceedings of the 13th Asian Logic Conference*, pages 1–28. World Sci. Publ., Hackensack, NJ, 2015.

[2] Cristian S. Calude. *Information and Randomness: An Algorithmic Perspective*, Second Edition, Revised and Extended, Springer-Verlag, Berlin, 2002.

[3] Cristian S. Calude and André Nies. Chaitin Ω numbers and strong reducibilities. *Journal of Universal Computer Science*, 3(11):1162–1166, 1997.

[4] Oswald Demuth and Antonín Kučera. Remarks on 1-genericity, semigenericity and related concepts. *Commentationes Mathematicae Universitatis Carolinae*, 28(1):85–94, 1987.

[5] Rodney G. Downey, Denis R. Hirschfeldt, Steffen Lempp and Reed Solomon. A Δ_2^0 set with no infinite low subset in either it or its complement. *The Journal of Symbolic Logic*, 66(3):1371–1381, 2001.

[6] Rodney G. Downey and Denis R. Hirschfeldt. *Algorithmic Randomness and Complexity*. Theory and Applications of Computability. Springer, New York, 2010.

[7] Johanna Franklin. Hyperimmune-free degrees and Schnorr triviality. *The Journal of Symbolic Logic*, 73:999–1008, 2008.

[8] Johanna Franklin and Frank Stephan. Schnorr trivial sets and truth-table reducibility. *The Journal of Symbolic Logic*, 75:501–521, 2010.

[9] Denis R. Hirschfeldt and Sebastiaan A. Terwijn. Limit computability and constructive measure. In *Computational Prospects of Infinity II: Presented Talks*, volume 15 of *Lecture Notes Series, Instute for Mathematical Sciences, National University of Singapore*, pages 131–141. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2008.

[10] Mushfeq Khan and Joseph S. Miller. Forcing with bushy trees. Manuscript, 2014.

[11] Bjørn Kjos-Hanssen, Wolfgang Merkle and Frank Stephan. Kolmogorov complexity and the recursion theorem. *Transactions of the American Mathematical Society*, 363(10):5465–5480, 2011.

[12] Bjørn Kjos-Hanssen, André Nies and Frank Stephan. Lowness for the class of Schnorr random reals. *SIAM Journal on Computing*, 35(3):647–657, 2005.

[13] Stuart Alan Kurtz. *Randomness and Genericity in the Degrees of Unsolvability.* ProQuest LLC, Ann Arbor, MI, 1981. Thesis (Ph.D.) – University of Illinois at Urbana-Champaign.

[14] Ming Li and Paul Vitányi. *An Introduction to Kolmogorov Complexity and Its Applications.* Third Edition, Springer, Heidelberg, 2008.

[15] Webb Miller and Donald A. Martin. The degrees of hyperimmune sets. *Zeitschrift für Mathematische Logik und Grundlagen der Mathematik*, 14:159–166, 1968.

[16] André Nies. *Computability and Randomness.* Oxford Science Publications, 2009.

[17] Piergiorgio Odifreddi. *Classical Recursion Theory*, volume 125 of *Studies in Logic and the Foundations of Mathematics.* North-Holland Publishing Co., Amsterdam, 1989.

[18] Piergiorgio Odifreddi. *Classical Recursion Theory II*, volume 143 of *Studies in Logic and the Foundations of Mathematics.* North-Holland Publishing Co., Amsterdam, 1999.

[19] Nicholas Rupprecht. Relativized Schnorr tests with universal behavior. *Archive for Mathematical Logic*, 49(5):555–570, 2010.

[20] Nicholas Rupprecht. *Effective correspondents to carcindal characteristics in Chichon's diagram.* PhD Thesis, University of Michigan, 2010.

[21] Robert I. Soare. *Recursively Enumerable Sets and Degrees.* Perspectives in Mathematical Logic. Springer-Verlag, Berlin, 1987. A study of computable functions and computably generated sets.

[22] Sebastiaan A. Terwijn. On the quantitative structure of Δ_2^0. In *Reuniting the antipodes—constructive and nonstandard views of the continuum (Venice, 1999)*, volume 306 of *Synthese Library*, pages 271–283. Kluwer Academic Publishers, Dordrecht, 2001.

[23] Sebastiaan A. Terwijn and Domenico Zambella. Algorithmic randomness and lowness. *The Journal of Symbolic Logic*, 66:1199–1205, 2001.