Midterm Examination 1
MA 3205: Set Theory

15.09.2009, 12.00-12.45h

Matriculation Number: _______________

Rules
Each question contains as many marks as it has subquestion. Each correct subquestion gives 1 mark. The maximum score is 15 marks.

Question 1. Determine the following sets where $A = \{1, 2, 4, 8, 16\}$ and $B = \{3, 4, 5, 6, 7, 8\}$:
(a) $A \cup B = \{\underline{\quad 1, 2, 3, 4, 5, 6, 7, 8, 16 \quad}\}$;
(b) $A \cap B = \{\underline{\quad 4, 8 \quad}\}$;
(c) $A \Delta B = \{\underline{\quad 1, 2, 3, 5, 6, 7, 16 \quad}\}$.
Here $\cup$ is the union, $\cap$ the intersection and Δ the symmetric difference.

Question 2. Let A be the powerset of $\mathbb{N}$, that is, let A be the set of all subsets of $\mathbb{N}$. Check the correct box for each set.

(a) The set $\{B \in A : \mathbb{N} \subseteq B\}$ is
☐ empty ☒ finite and not empty ☐ countable ☐ uncountable.

(b) The set $\{C \in A : C \text{ has 5 elements}\}$ is
☐ empty ☐ finite and not empty ☒ countable ☐ uncountable.

(c) The set $\{D \in A : D \text{ is infinite}\}$ is
☐ empty ☐ finite and not empty ☐ countable ☒ uncountable.

Question 3. (a) Is there a set A such that A has more elements then $\cup A$?

$\boxed{\text{x}}$ Yes; $\square$ No.

(b) Write a few lines to justify your answer (no complete proof needed, but it should make sense; only counted if (a) is correct).

Assume that A is the power set of another set, say $A = \mathcal{P}(\{0, 1\})$. Then $\cup A = \{0, 1\}$ has less elements than $A = \{\emptyset, \{0\}, \{1\}, \{0, 1\}\}$. In this example, $\cup A$ has 2 and A has 4 elements.

Question 4. (a) Is there a set B such that $B \neq \mathbb{N}$, B is transitive and B is inductive?

$\boxed{\text{x}}$ Yes; $\square$ No.

(b) Write a few lines to justify your answer (no complete proof needed, but it should make sense; only counted if (a) is correct).

There are many examples. One is V_ω which is inductive and transitive and a proper superset of $\mathbb{N}$. Another example is $\mathbb{N} \cup \{\mathbb{N}, S(\mathbb{N}), S(S(\mathbb{N})), S(S(S(\mathbb{N}))), \ldots\}$ where $S(X) = X \cup \{X\}$ for every set X.

Question 5. (a) Is there a set C such that the power set $\mathcal{P}(C)$ of C is countable?

$\square$ Yes; $\boxed{\text{x}}$ No.

Here recall that the statement "$\mathcal{P}(C)$ is countable" implies that "$\mathcal{P}(C)$ is infinite".

(b) Write a few lines to justify your answer (no complete proof needed, but it should make sense; only counted if (a) is correct).

If C is a finite set then $\mathcal{P}(C)$ is finite as well.

If C is infinite then $|C| \geq |\mathbb{N}|$. As the power set has always a cardinality larger than the set itself, it holds that $|\mathcal{P}(C)| > |C| \geq |\mathbb{N}|$ and $\mathcal{P}(C)$ is uncountable.

Question 6. (a) Determine all sets A which satisfy $\mathcal{P}(A) \subseteq \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}\}$:
$A = \emptyset$ satisfies $\mathcal{P}(A) = \{\emptyset\} \subseteq \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}\}$;
$A = \{\emptyset\}$ satisfies $\mathcal{P}(A) = \{\emptyset, \{\emptyset\}\} \subseteq \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}\}$;
$A = \{\{\emptyset\}\}$ satisfies $\mathcal{P}(A) = \{\emptyset, \{\{\emptyset\}\}\} \subseteq \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}\}$.
As the powerset of a set with at least 2 elements has at least 4 elements and as $A \in \mathcal{P}(A)$, there are no other examples.

(b) Determine all sets B which satisfy $B \subseteq \mathbb{N}$ and $\forall n[n \in B \Leftrightarrow n+2 \in B]$:
$B = \mathbb{N}$ and $B = \emptyset$ and $B = \{0, 2, 4, 6, \ldots\}$ and $B = \{1, 3, 5, 7, \ldots\}$; it was required to list all these four sets.

(c) How many sets $C \in \mathbb{N}$ have at most 5 elements?

☐ 0 ☐ 1 ☐ 2 ☐ 3 ☐ 4
☐ 5 ☒ 6 ☐ 7 ☐ infinitely many.

These six elements are $0, 1, 2, 3, 4, 5$ where $0 = \emptyset$, $1 = \{0\}$, $2 = \{0, 1\}$, ..., $5 = \{0, 1, 2, 3, 4\}$.