Midterm Examination 1  
MA 3205: Set Theory  
15.09.2009, 12.00-12.45h  
Matriculation Number: ____________

Rules  
Each question contains as many marks as it has subquestion. Each correct subquestion gives 1 mark. The maximum score is 15 marks.

**Question 1.** Determine the following sets where $A = \{1, 2, 4, 8, 16\}$ and $B = \{3, 4, 5, 6, 7, 8\}$:

(a) $A \cup B = \{1, 2, 3, 4, 5, 6, 7, 8, 16\}$;
(b) $A \cap B = \{4, 8\}$;
(c) $A \Delta B = \{1, 2, 3, 5, 6, 7, 16\}$.

Here $\cup$ is the union, $\cap$ the intersection and $\Delta$ the symmetric difference.

**Question 2.** Let $A$ be the powerset of $\mathbb{N}$, that is, let $A$ be the set of all subsets of $\mathbb{N}$. Check the correct box for each set.

(a) The set $\{B \in A : \mathbb{N} \subseteq B\}$ is
   - [ ] empty  [x] finite and not empty  [ ] countable  [ ] uncountable.

(b) The set $\{C \in A : C \text{ has 5 elements}\}$ is
   - [ ] empty  [ ] finite and not empty  [x] countable  [ ] uncountable.

(c) The set $\{D \in A : D \text{ is infinite}\}$ is
   - [ ] empty  [ ] finite and not empty  [ ] countable  [x] uncountable.
Question 3. (a) Is there a set $A$ such that $A$ has more elements than $\cup A$?

\[ \boxed{\text{Yes}}; \quad \boxed{\text{No}}. \]

(b) Write a few lines to justify your answer (no complete proof needed, but it should make sense; only counted if (a) is correct).

Assume that $A$ is the power set of another set, say $A = \mathcal{P}({0, 1})$. Then $\cup A = \{0, 1\}$ has less elements than $A = \{\emptyset, \{0\}, \{1\}, \{0, 1\}\}$. In this example, $\cup A$ has 2 and $A$ has 4 elements.

Question 4. (a) Is there a set $B$ such that $B \neq \mathbb{N}$, $B$ is transitive and $B$ is inductive?

\[ \boxed{\text{Yes}}; \quad \boxed{\text{No}}. \]

(b) Write a few lines to justify your answer (no complete proof needed, but it should make sense; only counted if (a) is correct).

There are many examples. One is $V_\omega$ which is inductive and transitive and a proper superset of $\mathbb{N}$. Another example is $\mathbb{N} \cup \{\mathbb{N}, S(\mathbb{N}), S(S(\mathbb{N})), S(S(S(\mathbb{N}))), \ldots\}$ where $S(X) = X \cup \{X\}$ for every set $X$.

Question 5. (a) Is there a set $C$ such that the power set $\mathcal{P}(C)$ of $C$ is countable?

\[ \boxed{\text{Yes}}; \quad \boxed{\text{No}}. \]

Here recall that the statement “$\mathcal{P}(C)$ is countable” implies that “$\mathcal{P}(C)$ is infinite”.

(b) Write a few lines to justify your answer (no complete proof needed, but it should make sense; only counted if (a) is correct).

If $C$ is a finite set then $\mathcal{P}(C)$ is finite as well.

If $C$ is infinite then $|C| \geq |\mathbb{N}|$. As the power set has always a cardinality larger than the set itself, it holds that $|\mathcal{P}(C)| > |C| \geq |\mathbb{N}|$ and $\mathcal{P}(C)$ is uncountable.
Question 6. (a) Determine all sets $A$ which satisfy $\mathcal{P}(A) \subseteq \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}\}$:

$A = \emptyset$ satisfies $\mathcal{P}(A) = \{\emptyset\} \subseteq \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}\}$;
$A = \{\emptyset\}$ satisfies $\mathcal{P}(A) = \{\emptyset, \{\emptyset\}\} \subseteq \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}\}$;
$A = \{\{\emptyset\}\}$ satisfies $\mathcal{P}(A) = \{\emptyset, \{\{\emptyset\}\}\} \subseteq \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}\}$.

As the powerset of a set with at least 2 elements has at least 4 elements and as $A \in \mathcal{P}(A)$, there are no other examples.

(b) Determine all sets $B$ which satisfy $B \subseteq \mathbb{N}$ and $\forall n \in B \iff n + 2 \in B$:

$B = \mathbb{N}$ and $B = \emptyset$ and $B = \{0, 2, 4, 6, \ldots\}$ and $B = \{1, 3, 5, 7, \ldots\}$; it was required to list all these four sets.

(c) How many sets $C \in \mathbb{N}$ have at most 5 elements?

\[ \square 0 \quad \square 1 \quad \square 2 \quad \square 3 \quad \square 4 \]
\[ \square 5 \quad \xmark 6 \quad \square 7 \quad \square \text{ininitely many}. \]

These six elements are 0, 1, 2, 3, 4, 5 where 0 = \emptyset, 1 = \{\emptyset\}, 2 = \{\emptyset, \{\emptyset\}\}, \ldots, 5 = \{0, 1, 2, 3, 4\}.$