Theory of Computation 6
Homomorphisms

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Repetition 1

Let $(\mathbf{Q_1}, \mathbf{\Sigma}, \delta_\mathbf{1}, \mathbf{s_1}, \mathbf{F_1})$ and $(\mathbf{Q_2}, \mathbf{\Sigma}, \delta_\mathbf{2}, \mathbf{s_2}, \mathbf{F_2})$ be dfas which recognise $\mathbf{L_1}$ and $\mathbf{L_2}$, respectively.

Consider $(\mathbf{Q_1} \times \mathbf{Q_2}, \mathbf{\Sigma}, \delta_\mathbf{1} \times \delta_\mathbf{2}, (\mathbf{s_1}, \mathbf{s_2}), \mathbf{F})$ with $(\delta_\mathbf{1} \times \delta_\mathbf{2})((\mathbf{q_1}, \mathbf{q_2}), \mathbf{a}) = (\delta_\mathbf{1}(\mathbf{q_1}, \mathbf{a}), \delta_\mathbf{2}(\mathbf{q_2}, \mathbf{a}))$. This automaton is called a product automaton and one can choose $\mathbf{F}$ such that it recognises the union or intersection or difference of the respective languages.

Union: $\mathbf{F} = \mathbf{F_1} \times \mathbf{Q_2} \cup \mathbf{Q_1} \times \mathbf{F_2}$;
Intersection: $\mathbf{F} = \mathbf{F_1} \times \mathbf{F_2} = \mathbf{F_1} \times \mathbf{Q_2} \cap \mathbf{Q_1} \times \mathbf{F_2}$;
Difference: $\mathbf{F} = \mathbf{F_1} \times (\mathbf{Q_2} - \mathbf{F_2})$;
Symmetric Difference: $\mathbf{F} = \mathbf{F_1} \times (\mathbf{Q_2} - \mathbf{F_2}) \cup (\mathbf{Q_1} - \mathbf{F_1}) \times \mathbf{F_2}$.

Repetition 2 and Gaps Filled

Regular languages are also closed under Kleene star, Kleene plus and concatenation: Use nfas for these and convert to dfas.

Context-free languages are closed under union, Kleene star, Kleene plus, concatenation and intersection with regular languages. They are in general not closed under intersection and complement.

Context-sensitive languages are closed under union, intersection, Kleene star, Kleene plus and concatenation. While these are easy to see, the following result is more difficult: They are also closed under complement (not part of this course).

Recursively enumerable languages are closed under union, intersection, Kleene star, Kleene plus and concatenation; they are not closed under complement.

Repetition 3: Palindromes

The members of the language $\{x \in \Sigma^* : x = x^{mi}\}$ are called palindromes. A palindrome is a word or phrase which looks the same from both directions.

An example is the German name “OTTO”; furthermore, when ignoring spaces and punctuation marks, a famous palindrome is the phrase “A man, a plan, a canal: Panama.” originating from the time when the canal in Panama was built.

The grammar with the rules $S \rightarrow aSa|aa|a|\varepsilon$ with a ranging over all members of Σ generates all palindromes; so for $\Sigma = \{0, 1, 2\}$ the rules of the grammar would be $S \rightarrow 0S0 \mid 1S1 \mid 2S2 \mid 00 \mid 11 \mid 22 \mid 0 \mid 1 \mid 2 \mid \varepsilon$.

The set of palindromes is not regular.

Homomorphism

Example

Let $\mathbf{ascii(Year\ 2019) = 5965617220323031 39}$ represent each letter of "$\mathbf{Year\ 2019}$" by its two-digit hexadecimal ASCII representation.

Definition 6.1

A homomorphism is a mapping $\mathbf{h}$ with domain $\mathbf{\Sigma}^*$ for some alphabet $\mathbf{\Sigma}$ which preserves concatenation: $\mathbf{h(v \cdot w) = h(v) \cdot h(w)}$.

Proposition 6.2

The homomorphism is determined by the images of the single letters and $\mathbf{h(w) = h(a_1) \cdot h(a_2) \cdot \ldots \cdot h(a_n)}$ for a word $\mathbf{w = a_1 a_2 \ldots a_n}$; $\mathbf{h}(\varepsilon) = \varepsilon$.

Quiz

What is $\mathbf{ascii(Year\ 1819)}$ for above homomorphism $\mathbf{ascii}$?

Exercises 6.3 and 6.4

Count the number of homomorphisms and list them; explain why there are not more. Two homomorphisms are the same iff they have the same values $\mathbf{h(0), h(1), h(2), h(3)}$. Here they take values from $\mathbf{4^*}$.

Exercise 6.3
How many homomorphisms $\mathbf{h}$ satisfy $\mathbf{h(012) = 44444}$, $\mathbf{h(102) = 444444}$, $\mathbf{h(00) = 44444}$ and $\mathbf{h(3) = 4}$?

Exercise 6.4
How many homomorphisms $\mathbf{h}$ satisfy $\mathbf{h(012) = 44444}$, $\mathbf{h(102) = 44444}$, $\mathbf{h(0011) = 444444}$ and $\mathbf{h(3) = 44}$?

Homomorphic Images

Theorem 6.5

The homomorphic images of regular and context-free languages are regular and context-free, respectively.

Construction

Given a homomorphism $\mathbf{h}$, replace in any rule of a given regular / context-free grammar every terminal $\mathbf{a}$ by the word $\mathbf{h(a)}$; these replacements only occur on the right side of the rules. The type of the grammar remains unchanged.

For a proof that $\mathbf{S} \Rightarrow^* \mathbf{w}$ in the original grammar iff $\mathbf{S} \Rightarrow \mathbf{h(w)}$ in the new grammar, one shows by induction for a derivation $\mathbf{S} \Rightarrow \mathbf{v_1} \Rightarrow \ldots \Rightarrow \mathbf{v_n} \Rightarrow \mathbf{w}$ translates into $\mathbf{h(S)} \Rightarrow \mathbf{h(v_1)} \Rightarrow \ldots \Rightarrow \mathbf{h(v_n)} \Rightarrow \mathbf{h(w)}$ where $\mathbf{h}$ is extended by letting $\mathbf{h(A)} = \mathbf{A}$ for all non-terminals $\mathbf{A}$. The converse also holds.

Example 6.6

One can apply the homomorphisms also directly to regular expressions using the rules $\mathbf{h(L \cup H) = h(L) \cup h(H)}$, $\mathbf{h(L \cdot H) = h(L) \cdot h(H)}$ and $\mathbf{h(L^*) = (h(L))^*}$. Thus one can move a homomorphism into the inner parts (which are the finite sets used in the regular expression) and then apply the homomorphism there.

So for the language $\mathbf{(\{0,1\}^* \cup \{0,2\}^*) \cdot \{33\}^*}$ and the homomorphism which maps each symbol $\mathbf{a}$ to $\mathbf{aa}$, one obtains the language $\mathbf{(\{00,11\}^* \cup \{00,22\}^*) \cdot \{3333\}^*}$.

Homomorphisms and Growth

Exercise 6.7

Consider the following statements for regular languages $\mathbf{L}$:

(a) $\mathbf{h}(\emptyset) = \emptyset$;

(b) If $\mathbf{L}$ is finite so is $\mathbf{h}(\mathbf{L})$;

(c) If $\mathbf{L}$ has polynomial growth so has $\mathbf{h}(\mathbf{L})$;

(d) If $\mathbf{L}$ has exponential growth so has $\mathbf{h}(\mathbf{L})$.

Which of these statements are true and which are false? Prove the answers. Use the following rules: Example 6.6; $\mathbf{H}^*$ has polymomial growth iff $\mathbf{H} \subseteq \{\mathbf{u}\}^*$ for some word $\mathbf{u}$; if $\mathbf{H}, \mathbf{K}$ have polynomial growth so do $\mathbf{H} \cup \mathbf{K}$ and $\mathbf{H} \cdot \mathbf{K}$.

Exercise 6.8

Construct a context-sensitive language $\mathbf{L}$ and a homomorphism $\mathbf{h}$ such that $\mathbf{L}$ has polynomial growth and $\mathbf{h}(\mathbf{L})$ has exponential growth.

Homomorphism Reduce Kleene star

One can reduce the number of stars in $\bigcup_{\mathbf{a} \in \mathbf{\Sigma}} \mathbf{aa}^*$ to two using intersection:

$$\mathbf{00}^* \cup \mathbf{11}^* \cup \mathbf{22}^* \cup \mathbf{33}^* = (\{\mathbf{0}, \mathbf{1}, \mathbf{2}, \mathbf{3}\} \cdot \{\mathbf{00}, \mathbf{11}, \mathbf{22}, \mathbf{33}\}^* \cdot \{\varepsilon, \mathbf{0}, \mathbf{1}, \mathbf{2}, \mathbf{3}\}) \cap (\{\mathbf{00}, \mathbf{11}, \mathbf{22}, \mathbf{33}\}^* \cdot \{\varepsilon, \mathbf{0}, \mathbf{1}, \mathbf{2}, \mathbf{3}\}).$$

The general results needs also a homomorphism.

Theorem 6.9
Let $\mathbf{L}$ be a regular language. Then there are two regular expressions σ, τ each containing only one Kleene star and some finite sets and concatenations and there is one homomorphism $\mathbf{h}$ such that $\mathbf{L}$ is described by $\mathbf{h}(\sigma \cap \tau)$.

The idea is to encode states of a dfa into the symbols; expressions σ and τ test state-transitions at even and odd positions, respectively; $\mathbf{h}$ removes the state markers from the symbols.

Construction

Let $(\mathbf{Q}, \mathbf{\Sigma}, \delta, \mathbf{s}, \mathbf{F})$ be a dfa recognising the language and let $\mathbf{\Gamma} = \mathbf{Q} \times \mathbf{\Sigma}$ and

$$
\begin{aligned}
\mathbf{\Gamma_1} &= \{(\mathbf{q}, \mathbf{a})(\mathbf{p}, \mathbf{b}) \in \mathbf{\Gamma} \times \mathbf{\Gamma} : \delta(\mathbf{q}, \mathbf{a}) = \mathbf{p}\}; \\
\mathbf{\Gamma_2} &= \{(\mathbf{q}, \mathbf{a})(\mathbf{p}, \mathbf{b}) \in \mathbf{\Gamma_1} : \delta(\mathbf{p}, \mathbf{b}) \in \mathbf{F}\}; \\
\mathbf{\Gamma_3} &= \{(\mathbf{q}, \mathbf{a}) : \delta(\mathbf{q}, \mathbf{a}) \in \mathbf{F}\}; \\
\mathbf{\Gamma_4} &= \{\varepsilon : \mathbf{s} \in \mathbf{F}\}; \\
\mathbf{\Gamma_5} &= \{(\mathbf{s}, \mathbf{a}) : \mathbf{a} \in \mathbf{\Sigma}\}.
\end{aligned}
$$

The expression is $\mathbf{h}(\sigma \cap \tau)$ where $\mathbf{h}((\mathbf{q}, \mathbf{a})) = \mathbf{a}$;

$$
\begin{aligned}
\sigma &= (\mathbf{\Gamma_1^*} \cdot (\mathbf{\Gamma_2} \cup \mathbf{\Gamma_3}) \cup \mathbf{\Gamma_4}); \\
\tau &= (\mathbf{\Gamma_5} \cdot \mathbf{\Gamma_1^*} \cdot (\mathbf{\Gamma} \cup \{\varepsilon\}) \cup \mathbf{\Gamma_4}).
\end{aligned}
$$

Odd transitions and acceptance checked by σ;
Even transitions and start checked by τ.

Context-Senstive Languages

Theorem 6.11

Every recursively enumerable language (= language generated by some grammar) is the homomorphic image of a context-sensitive language.

The idea is that if some grammar generates $(\mathbf{N}, \{\mathbf{1}, \mathbf{2}, \ldots, \mathbf{k}\}, \mathbf{P}, \mathbf{S})$ for $\mathbf{L}$, one can make a new grammar for a context-sensitive language $\mathbf{H}$ such that for all $\mathbf{w} \in \{\mathbf{1}, \mathbf{2}, \ldots, \mathbf{k}\}^*$, $\mathbf{w} \in \mathbf{L}$ iff $\mathbf{w} \cdot \mathbf{0}^\ell \in \mathbf{H}$ for some ℓ. These additional $\mathbf{0}$ will be used to make words longer so that in the new grammar, all rules $\mathbf{l} \to \mathbf{r}$ satisfy $|\mathbf{l}| \leq |\mathbf{r}|$ which is obtained sufficiently many $\mathbf{0}$ on the ride side and by making rules for $\mathbf{0}$ to swap with other symbols to move right.

Images of Homomorphisms

Determine $\mathbf{h(L)}$ for the following languages:

(a) $\{\mathbf{0}, \mathbf{1}, \mathbf{2}\}^*$;

(b) $\{\mathbf{00}, \mathbf{11}, \mathbf{22}\}^* \cap \{\mathbf{000}, \mathbf{111}, \mathbf{222}\}^*$;

(c) $(\{\mathbf{00}, \mathbf{11}\}^* \cup \{\mathbf{00}, \mathbf{22}\}^* \cup \{\mathbf{11}, \mathbf{22}\}^*) \cdot \{\mathbf{011222}\}$;

(d) $\{\mathbf{w} \in \{\mathbf{0}, \mathbf{1}\}^* : \mathbf{w}$ has more $\mathbf{1}$s than it has $\mathbf{0}$s$\}$.

Exercise 6.13
$\mathbf{h}$ is given as $\mathbf{h(0) = 1}$, $\mathbf{h(1) = 22}$, $\mathbf{h(2) = 333}$.

Exercise 6.14
$\mathbf{h}$ is given as $\mathbf{h(0) = 3}$, $\mathbf{h(1) = 4}$, $\mathbf{h(2) = 334433}$.

Exercise 6.15

Let a homomorphism $\mathbf{h} : \{\mathbf{0}, \mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}, \mathbf{5}, \mathbf{6}, \mathbf{7}, \mathbf{8}, \mathbf{9}\}^* \rightarrow \{\mathbf{0}, \mathbf{1}, \mathbf{2}, \mathbf{3}\}^*$ be given by the equations $\mathbf{h}(\mathbf{0}) = \mathbf{0}$, $\mathbf{h}(\mathbf{1}) = \mathbf{h}(\mathbf{4}) = \mathbf{h}(\mathbf{7}) = \mathbf{1}$, $\mathbf{h}(\mathbf{2}) = \mathbf{h}(\mathbf{5}) = \mathbf{h}(\mathbf{8}) = \mathbf{2}$, $\mathbf{h}(\mathbf{3}) = \mathbf{h}(\mathbf{6}) = \mathbf{h}(\mathbf{9}) = \mathbf{3}$. Interpret the images of $\mathbf{h}$ as quarternary numbers (numbers of base four, so $\mathbf{12321}$ represents $\mathbf{1}$ times two hundred fifty six plus $\mathbf{2}$ times sixty four plus $\mathbf{3}$ times sixteen plus $\mathbf{2}$ times four plus $\mathbf{1}$). Prove the following:

- Every quarternary number is the image of a decimal number without leading zeroes;
- A decimal number $\mathbf{w}$ has leading zeroes iff the quarternary number $\mathbf{h}(\mathbf{w})$ has leading zeroes;
- A decimal number $\mathbf{w}$ is a multiple of three iff the quarternary number is a multiple of three.

Exercise 6.16

Consider only homomorphisms
$\mathbf{h} : \{\mathbf{0},\mathbf{1},\mathbf{2},\mathbf{3},\mathbf{4},\mathbf{5},\mathbf{6},\mathbf{7},\mathbf{8},\mathbf{9}\}^* \to \{\mathbf{0},\mathbf{1}\}^*$ such that

- $\mathbf{h}(\mathbf{w})$ has leading zeroes iff $\mathbf{w}$ has;
- $\mathbf{h}(\mathbf{0}) = \mathbf{0}$;
- the range of $\mathbf{h}$ is $\{\mathbf{0},\mathbf{1}\}^*$.

For each of $\mathbf{p} = \mathbf{2},\mathbf{3},\mathbf{5}$, answer the following question: Can one choose $\mathbf{h}$ such that, in addition, $\mathbf{w}$ is a multiple of $\mathbf{p}$ iff $\mathbf{h}(\mathbf{w})$ is as a binary number, is a multiple of $\mathbf{p}$?

If $\mathbf{h}$ can be chosen as desired then list this $\mathbf{h}$ else prove that such a homomorphism $\mathbf{h}$ cannot exist.

Exercise 6.17

Construct a homomorphism $\mathbf{h} : \{\mathbf{0}, \mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}, \mathbf{5}, \mathbf{6}, \mathbf{7}, \mathbf{8}, \mathbf{9}\}^* \to \{\mathbf{0}, \mathbf{1}\}^*$ such that for every $\mathbf{w}$ the number $\mathbf{h}(\mathbf{w})$ has never leading zeroes and the remainder of the decimal number $\mathbf{w}$ when divided by nine is the same as the remainder of the binary number $\mathbf{h}(\mathbf{w})$ when divided by nine.

Note that here it is not required that the range covers all binary numbers.

Fibonacci Representation

Let $a_0 = 1$, $a_1 = 1$, $a_2 = 2$ and, for all n, $a_{n+2} = a_n + a_{n+1}$. Every number is the sum of non-neighbouring Fibonacci numbers: For each non-zero n there is a unique $b_m b_{m-1} \ldots b_0 \in (10^+)^+$ with

$$n = \sum_{k=0,1,\ldots,m} b_k \cdot a_k.$$

So 1010 represents four and 100100 represents ten.

Exercise 6.18: Construct a homomorphism $h : \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\} \to \{0, 1\}^*$ such that $h(0) = 0$ and the image of all decimal numbers is the regular set $\{0\} \cup (10^+)^+$. Show that all h satisfying this also satisfy the following statement: For every $p > 1$ there is a decimal number w such that (w is a multiple of p iff $h(w)$ is not a multiple of p).

Inverse Homomorphism

Description 6.19

Let $\mathbf{h}$ have domain $\mathbf{\Sigma}^*$ and the set $\mathbf{h^{-1}(L) = \{w \in \Sigma^* : h(w) \in L\}}$ is called the inverse image of $\mathbf{h}$. $\mathbf{h^{-1}}$ satisfies the following rules:

(a) $\mathbf{h^{-1}(L) \cap h^{-1}(H) = h^{-1}(L \cap H)}$;

(b) $\mathbf{h^{-1}(L) \cup h^{-1}(H) = h^{-1}(L \cup H)}$;

(c) $\mathbf{h^{-1}(L) \cdot h^{-1}(H) \subseteq h^{-1}(L \cdot H)}$;

(d) $\mathbf{h^{-1}(L)^* \subseteq h^{-1}(L^*)}$.

Theorem 6.20 and Exercise 6.21

Theorem 6.20
If $\mathbf{L}$ is on level $\mathbf{k}$ of the Chomsky hierarchy and $\mathbf{h}$ is an homomorphism then $\mathbf{h^{-1}(L)}$ is on level $\mathbf{k}$ of the Chomsky hierarchy.

Construction for the regular case: If $(\mathbf{Q}, \mathbf{\Gamma}, \gamma, \mathbf{s}, \mathbf{F})$ is a dfa recognising $\mathbf{L}$ and $\mathbf{h} : \mathbf{\Sigma}^* \to \mathbf{\Gamma}^*$ is an homomorphism then $(\mathbf{Q}, \mathbf{\Sigma}, \delta, \mathbf{s}, \mathbf{F})$ is a dfa recognising $\mathbf{h^{-1}(L)}$ where, for every $\mathbf{q} \in \mathbf{Q}$ and $\mathbf{a} \in \mathbf{\Sigma}$, $\delta(\mathbf{q}, \mathbf{a}) = \gamma(\mathbf{q}, \mathbf{h}(\mathbf{a}))$.

Exercise 6.21
Let $\mathbf{h} : \{\mathbf{0}, \mathbf{1}, \mathbf{2}, \mathbf{3}\}^* \to \{\mathbf{0}, \mathbf{1}, \mathbf{2}, \mathbf{3}\}^*$ be given by $\mathbf{h(0) = 00}$, $\mathbf{h(1) = 012}$, $\mathbf{h(2) = 123}$ and $\mathbf{h(3) = 1}$ and let $\mathbf{L}$ consist of all words containing exactly five $\mathbf{0}$s and at least one $\mathbf{2}$. Construct a complete dfa recognising $\mathbf{h^{-1}(L)}$.

Generalised Homomorphism

Description 6.22

A generalised homomorphism is a mapping from regular sets to regular sets which satisfies $\mathbf{h(L \cup H) = h(L) \cup h(H)}$, $\mathbf{h(L \cdot H) = h(L) \cdot h(H)}$, $\mathbf{h(L^*) = (h(L))^*}$ and $\mathbf{h(\emptyset) = \emptyset}$ for all regular sets $\mathbf{L}$ and $\mathbf{H}$.

Examples 6.23

The following mappings are generalised homomorphisms:

- $\mathbf{L \mapsto L \cap \{\varepsilon\}}$;
- $\emptyset \mapsto \emptyset$ and $\mathbf{L} \mapsto \{\varepsilon\}$ for all non-empty sets $\mathbf{L}$;
- $\emptyset \mapsto \emptyset$, $\{\varepsilon\} \mapsto \{\varepsilon\}$ and $\mathbf{L \mapsto \Sigma^*}$ for all other sets $\mathbf{L}$;
- $\mathbf{L \mapsto L}$ (identity mapping);
- $\mathbf{L \mapsto \{v \in \Sigma^* : \exists w \in L\, [|v| = |w|]\}}$.

Exercises 6.24-6.25

Exercise 6.24

Show that whenever $\mathbf{h} : \mathbf{\Sigma}^* \to \mathbf{\Gamma}^*$ is a homomorphism then the mapping $\mathbf{L} \mapsto \{\mathbf{h}(\mathbf{u}) : \mathbf{u} \in \mathbf{L}\}$ is a generalised homomorphism which maps regular subsets of $\mathbf{\Sigma}^*$ to regular subsets of $\mathbf{\Gamma}^*$.

Exercise 6.25

Let $\mathbf{h}$ be any given generalised homomorphism. Show by structural induction that $\mathbf{h}(\mathbf{L}) = \bigcup_{\mathbf{u} \in \mathbf{L}} \mathbf{h}(\mathbf{u})$ for all regular languages $\mathbf{L}$. Furthermore, show that every mapping $\mathbf{h}$ satisfying $\mathbf{h}(\{\varepsilon\}) = \{\varepsilon\}$, $\mathbf{h}(\mathbf{L}) = \bigcup_{\mathbf{u} \in \mathbf{L}} \mathbf{h}(\{\mathbf{u}\})$ and $\mathbf{h}(\mathbf{L} \cdot \mathbf{H}) = \mathbf{h}(\mathbf{L}) \cdot \mathbf{h}(\mathbf{H})$ for all regular subsets $\mathbf{L}, \mathbf{H}$ of $\mathbf{\Sigma}^*$ is a generalised homomorophism. Is the same true if one weakens the condition $\mathbf{h}(\{\varepsilon\}) = \{\varepsilon\}$ to $\varepsilon \in \mathbf{h}(\{\varepsilon\})$?

Exercises 6.26-6.28

Exercise 6.26

Construct a mapping which satisfies $\mathbf{h}(\emptyset) = \emptyset$, $\mathbf{h}(\{\varepsilon\}) = \{\varepsilon\}$, $\mathbf{h}(\mathbf{L} \cup \mathbf{H}) = \mathbf{h}(\mathbf{L}) \cup \mathbf{h}(\mathbf{H})$ and $\mathbf{h}(\mathbf{L} \cdot \mathbf{H}) = \mathbf{h}(\mathbf{L}) \cdot \mathbf{h}(\mathbf{H})$ for all regular languages $\mathbf{L}, \mathbf{H}$ but which does not satisy $\mathbf{h}(\mathbf{L}) = \bigcup_{\mathbf{u} \in \mathbf{L}} \mathbf{h}(\{\mathbf{u}\})$ for some infinite regular set $\mathbf{L}$.

Exercise 6.27

Assume that $\mathbf{h}$ is a generalised homomorphism and $\mathbf{k}(\mathbf{L}) = \mathbf{h}(\mathbf{L}) \cdot \mathbf{h}(\mathbf{L})$. Is $\mathbf{k}$ a generalised homomorphism? Prove the answer.

Exercise 6.28

Assume that $\mathbf{h}$ is a generalised homomorphism and $\ell(\mathbf{L}) = \bigcup_{\mathbf{u} \in \mathbf{h}(\mathbf{L})} \mathbf{\Sigma}^{|\mathbf{u}|}$, where $\mathbf{\Sigma}^{\mathbf{0}} = \{\varepsilon\}$. Is ℓ a generalised homomorphism? Prove the answer.

Exercise 6.29

Let $\mathbf{\Sigma = \{0, 1, 2\}}$ and $\mathbf{h}$ be the generalised homomorphism given by $\mathbf{h(\{0\}) = \{1, 2\}}$, $\mathbf{h(\{1\}) = \{0, 2\}}$ and $\mathbf{h(\{2\}) = \{0, 1\}}$. Which of the following statements are true for this $\mathbf{h}$ and all regular subsets $\mathbf{L, H}$ of $\mathbf{\Sigma^*}$:

(a) If $\mathbf{L \neq H}$ then $\mathbf{h(L) \neq h(H)}$;
(b) If $\mathbf{L \subseteq H}$ then $\mathbf{h(L) \subseteq h(H)}$;
(c) If $\mathbf{L}$ is finite then $\mathbf{h(L)}$ is finite;
(d) If $\mathbf{L}$ is infinite then $\mathbf{h(L)}$ is infinite and has exponential growth.

Prove the answers. The formula $\mathbf{h(L) = \bigcup_{u \in L} h(\{u\})}$ from Exercise 6.25 can be used without proof for this exercise.

Generalised Homomorphisms

Determine $\mathbf{h(L) = \bigcup_{a_1 a_2 \ldots a_n \in L} h(a_1) \cdot h(a_2) \cdot \ldots \cdot h(a_n)}$ for the following languages; if possible give regular expressions.

(a) $\mathbf{\{00, 01, 02, 10, 11, 12, 20, 21, 22\}^*}$;

(b) $\mathbf{\{00, 11, 22\}^* \cdot \{000, 111, 222\}}$;

(c) $\mathbf{\{0^n 1^n 2^n : n \geq 2\}}$.

Exercise 6.30
$\mathbf{h}$ is given as $\mathbf{h(0) = \{3, 4\}^+}$, $\mathbf{h(1) = \{3, 5\}^+}$, $\mathbf{h(2) = \{4, 5\}^+}$.

Exercise 6.31
$\mathbf{h}$ is given as $\mathbf{h(0) = \{\varepsilon, 3, 33\}}$, $\mathbf{h(1) = \{\varepsilon, 4, 44\}}$,
$\mathbf{h(2) = \{\varepsilon, 5, 55\}}$.

Exercise 6.32
$\mathbf{h}$ is given as $\mathbf{h(a) = \{aaa, aaaa\}^+}$ for all letters $\mathbf{a \in \{0, 1, 2\}}$.