Rules: This test carries 20 marks and consists of 4 questions. Each questions carries 5 marks; full marks for a correct solution; a partial solution can give a partial credit. Use the backside of the page if the space for a question is insufficient.

Question 1 [5 marks]

Construct a complete deterministic finite automaton with as few states as possible which recognises the set \( ((\{00\} \cdot \{0\})^* \cup (\{11\} \cdot \{1\})^* \cup (\{22\} \cdot \{2\})^*)^* \). The alphabet is \( \{0, 1, 2\} \). Recall that a finite automaton is deterministic and complete iff for every state \( q \) and every symbol \( a \) there is exactly one successor state \( \delta(q, a) \) to which it can go.

Solution. The dfa is given as follows: The set of states is \( \{s, q_0, q_1, q_2, r_0, r_1, r_2, p\} \) and the alphabet is \( \{0, 1, 2\} \). The state-transition function \( \delta \) uses in the following definition a symbol \( a \) (if needed) as the index of the state and \( b \) as the input symbol currently processed: \( \delta(s, b) = q_b \); if \( a = b \) then \( \delta(q_a, b) = r_a \) else \( \delta(q_a, b) = p \); if \( a = b \) then \( \delta(r_a, b) = r_a \) else \( \delta(r_a, b) = q_b \); \( \delta(p, b) = p \). Furthermore \( s \) is the starting state and \( \{s, r_a, r_b, r_c\} \) is the set of accepting states. The states \( q_a \) differ from all \( r_b \) as the first are rejecting and the latter are accepting. The start state \( s \) is the unique state from which one can go within two but not within one step into an accepting state. The state \( p \) is the unique state from which one cannot go into an accepting state. Note that if \( a \neq b \) then the states \( q_a \) and \( q_b \) differ as from \( q_a \) one goes on \( a \) into an accepting state but not from \( q_b \); similarly for \( r_a \) and \( r_b \). Furthermore, all states are reachable. Thus the dfa is minimal.
Question 2 [5 marks]

Recall the traditional form of the Pumping Lemma: Let \( L \subseteq \Sigma^* \) be an infinite regular language. Then there is a constant \( k \) such that for every \( u \in L \) of length at least \( k \) there is a representation \( x \cdot y \cdot z = u \) such that \( |xy| \leq k \), \( y \neq \varepsilon \) and \( xy^*z \subseteq L \).

Recall that a word \( w \) is a palindrome iff the mirror image of \( w \) is equal to \( w \); so 00110001100 and 01210 are palindromes while 0001 is not. Let \( H = \{w \in 0^+1^+2^+1^+0^+: w \text{ is a palindrome}\} \). Which of the following three choices is correct?

(a) \( H \) is regular and satisfies the Pumping Lemma;

(b) \( H \) is not regular but still satisfies the Pumping Lemma;

(c) \( H \) does not satisfy the Pumping Lemma and is thus not regular.

Prove your answer.

**Solution.** The correct choice is (c).

Assume that \( H \) satisfies the Pumping Lemma with constant \( k \) and consider the word \( 0^k1^k2^k1^k0^k \) which is in \( H \). If \( H \) would be regular then there are \( x, y, z \) with \( xyz = 0^k1^k2^k1^k0^k \), \( y \neq \varepsilon \), \( |xy| \leq k \) and \( xz \in H \). Due to the length constraints, \( x \in 0^* \) and \( y \in 0^+ \). Now \( xz = 0^h1^k2^k1^k0^k \) for a number \( h < k \) and is not a palindrome, thus \( xz \notin H \) and the Pumping Lemma cannot be satisfied for \( H \). Thus \( H \) cannot be a regular set as well.
Question 3 [5 marks] 

Consider the context-free grammar

\[ \{S, T\}, \{0, 1, 2\}, \{S \rightarrow T1T1T|T2T2T, T \rightarrow TT|0|1|2|\varepsilon\}, S\].

Is the language \(L\) generated by this grammar regular? If so, provide a non-deterministic finite automaton recognising \(L\); if not, give a proof that the language is not regular.

**Solution.** The language is regular. A non-deterministic automaton for this language is given as follows:

![Automaton Diagram](https://via.placeholder.com/150)
Question 4 [5 marks]  

The following \{0,1,2,3\}-valued function $F$ is defined by structural induction for all regular expressions:

- $F(\emptyset) = 0$, $F(\{\varepsilon\}) = 1$;
- $F(\{w_1, w_2, \ldots, w_n\}) = 2$ in the case that at least one of the $w_m$ is a nonempty word – otherwise the previous case applies;
- $F((\sigma \cup \tau)) = \max\{F(\sigma), F(\tau)\}$;
- If $F(\sigma) = 0$ or $F(\tau) = 0$ then $F((\sigma \cdot \tau)) = 0$ else $F((\sigma \cdot \tau)) = \max\{F(\sigma), F(\tau)\}$;
- If $F(\sigma) \leq 1$ then $F(\sigma^*) = 1$ else $F(\sigma^*) = 3$.

In these definitions, it is always assumed that brackets are used to make the breaking down of expressions unique and that $\sigma, \tau$ are valid regular expressions using as constants $\emptyset$ and lists of finite sets of strings and as connectives $\cup, \cdot$ and $\ast$. Answer the following questions:

- What is $F(((\{00,11\}^* \cdot \emptyset) \cup \{00,11,22\}))$?
- For which regular expressions does it hold that $F(\sigma) = 3$?
- Are there two different regular expressions $\sigma, \tau$ describing the same set such that $F(\sigma) \neq F(\tau)$?

Give short explanations for your answers.
Solution. $F(((\{00, 11\} \cdot \emptyset) \cup \{00, 11, 22\})) = \max\{F(((\{00, 11\} \cdot \emptyset)), F(\{00, 11, 22\})\}$
$= \max\{F(\emptyset), F(\{00, 11, 22\})\} = \max\{0, 2\} = 2$. In general, the function $F$ of an expression $\sigma$ for a set $L$ is satisfies the following equation:

$$F(\sigma) = \begin{cases} 
0 & \text{if } L = \emptyset; \\
1 & \text{if } L = \{\varepsilon\}; \\
2 & \text{if } L \text{ is finite and contains a nonempty word}; \\
3 & \text{if } L \text{ is infinite.}
\end{cases}$$

In particular, if $\sigma, \tau$ describe the same set then $F(\sigma) = F(\tau)$ and $F(\sigma) = 3$ iff $\sigma$ describes an infinite set. One can verify above equation on $F$ by induction: The conditions are hard-coded for lists of members of finite sets.

If $\sigma = (\tau \cup \rho)$ then $F(\sigma) = 0$ iff both $F(\tau), F(\rho) = 0$ iff both $\tau, \rho$ describe the empty set so that $\sigma$ also describes the empty set; similarly $\sigma$ describes $\{\varepsilon\}$ iff one of $\tau, \rho$ describes the set $\{\varepsilon\}$ and the other one either the same set or the empty set, so $F(\sigma) = 1$ iff $\max\{F(\tau), F(\rho)\} = 1$; $F(\sigma) = 2$ iff $\sigma$ describes a finite set containing a non-empty string iff one of $\rho, \tau$ does and if both sets are finite iff $\max\{F(\rho), F(\tau)\} = 2$; $F(\sigma) = 3$ iff one of $\rho, \tau$ describe an infinite set iff at least one of $F(\rho), F(\tau)$ is 3.

Similarly one can verify the rules for $\sigma = \rho \cdot \tau$ with the special case in mind that the concatenation with an empty set gives the empty set.

Furthermore $F(\tau^*) = 3$ iff $\tau^*$ describes an infinite set iff $\tau$ contains a nonempty string iff $F(\tau) \geq 2$; $F(\tau^*) = 1$ iff $\tau^*$ describes the set $\{\varepsilon\}$ iff $\tau$ describes either $\emptyset$ or $\{\varepsilon\}$ iff $F(\tau) \leq 1$. 

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