Some applications of recursion theory to geometric measure theory

Liang Yu

Nanjing University

31 Jan. 2024

AD

Let *AD* be the axiom of determinacy.

Definition

Strong Turing determinacy (sTD) says that for every set *A* of *reals* ranging Turing degrees cofinally, *A* has a pointed subset.

AD

Theorem (Martin)

Over ZF, $AD \rightarrow sTD$.

Geometric measure theory (1)

Given a non-empty $U \subseteq \mathbb{R}$, the *diameter* of U is

$$diam(U) = |U| = \sup\{|x - y| : x, y \in U\}.$$

Given any set $E \subseteq \mathbb{R}$ and $d \geq 0$, let

$$\mathcal{H}^d(E) = \lim_{\delta \to 0} \inf\{\sum_{i<\omega} |U_i|^d : \{U_i\} \text{ is an open cover of } E \wedge \forall i\ |U_i| < \delta\},$$

$\mathcal{P}_0^d(E) = \lim_{\delta\to 0} \sup\{\sum_{i<\omega} |B_i|^d :$
$\{B_i\}$ is a collection of disjoint balls of radii at most δ with centres in $E\}$
and

$$\mathcal{P}^d(E) = \inf\{\sum_{i<\omega} \mathcal{P}_0^d(E_i) \mid E \subseteq \bigcup_{i<\omega} E_i\}.$$

Geometric measure theory (2)

Definition

Given any set E,

- the *Hausdorff dimension* of E, or $\mathrm{Dim}_H(E)$, is
$$\inf\{d \mid \mathcal{H}^d(E) = 0\};$$
- the *Packing dimension* of E, or $\mathrm{Dim}_P(E)$, is
$$\inf\{d \mid \mathcal{P}^d(E) = 0\}.$$

Besicovitch and Davis theorem

Theorem (Besicovitch and Davis)

For any analytic set A, $\mathrm{Dim}_H(A) = \sup_{F \subseteq A \wedge F \text{ is closed}} \mathrm{Dim}_H(F)$.

Point-to-set principle and its applications

Let $\mathtt{dim}_H^x(y) = \underline{\lim}_{n\to\infty} \frac{K^x(y\upharpoonright n)}{n}$.

Theorem (Lutz and Lutz)

For any set A of reals, $\mathtt{Dim}_H(A) = \min_x \max_{r\in A} \mathtt{dim}_H^x(r)$.

Theorem (Slaman)

Assume that $V = L$, then BD-theorem fails for a Π_1^1-set.

One may slightly weaken the assumption to be "$(\mathbb{R})^L$ is not null".

Low for Hausdorff dimension

Theorem (Lempp, Miller, Ng, Turetsky and Weber)

For any real x, there is a real y low for Hausdorff dimension but $y' \geq_T x$.

BD-theorem under AD

Theorem (Peng, Wu and Y; Crone, Fishman and Jackson proves the consequence under $ZF + DC + AD$.)

Assume that $ZF + sTD$, BD-theorem holds for every set of reals.

Proof.

Fix any nonempty set A. For the simplicity, we may assume that $\mathrm{Dim}_H(A) = 1$.
By the results above, there is some e so that $B = \{x \mid \Phi_e^{x'} \in A$ has effective Hausdorff dimension 1 relative to $x\}$ ranges Turing degrees cofinally. By sTD, B has a pointed subset P.
Then $C = \{r \mid \exists x \in P \Phi_e^{x'} = r\}$ is an analytic subset of A with Hausdorff dimension 1. □

More results

Theorem (Joyce and Preiss)

For any analytic set A, $\mathrm{Dim}_P(A) = \sup_{F \subseteq A \wedge F \text{ is closed}} \mathrm{Dim}_P(F)$.

By a similar method, one may show that Joyce-Preiss theorem holds for arbitrary set under $ZF + sTD$. Note that Slaman's result holds for the packing dimension.

Some questions

Question

1. *What is the consistency strength of BD- and JP-theorems for arbitrary sets?*
2. *What is the consistency strength that every set of Turing degrees is measurable?*

Lutz-Lutz's theorem revisited (1)

Theorem

$\mathcal{H}^d(A) = 0$ *if and only if there is a real* x *and a constant* c *so that* $\forall z \in A \exists^\infty n (K^x(z \upharpoonright n) \leq dn + c)$.

Proof.

If direction, by Chaitin's counting theorem $(|\{\sigma \in 2^n \mid K(\sigma) \leq n + K(n) - d\}| \leq 2^{n-d+c_0})$.
Only if direction, by Chaitin-Kraft theorem. □

Two functions

Let

$$L_0(k) = \min\{n \mid \forall m \geq n K(m) \geq k\}$$
$$L_1(k) = \min\{n \mid \sum_{m \geq n} 2^{-K(m)} \leq 2^{-k}\}.$$

Lemma

$\exists c \forall k (L_1(k) \leq L_0(k+c))$.

Proof.

Clearly $L_1(k) \geq L_0(k)$. Suppose that $\forall c \exists k L_1(k) - L_0(k) > c$. By the recursion theorem, there is a prefix-pre machine M_e: for any k, if $K(m) < k + e + 1$, then $\sum_{m' > m} 2^{-K(m')} > 2^{-k}$. Define $K_M(m) = k$. So $L_0(k+e) \geq L_0^M(k) \geq m \geq L_0(k+e+1)$, a contradiction. □

Lutz-Lutz's theorem revisited (2)

Theorem

$\mathcal{H}^d(A) < \infty$ if and only if there is some c so that $\exists x \exists^\infty k \forall z \in A \exists n > L_0^x(k)(K^x(z \upharpoonright n) < dn + k + c)$.

Proof.

If direction, by Chaitin's counting theorem and the Lemma.
Only if direction, by building a machine. □

Another Besicovitch-Davis theorem

A set A is σ-finite for $\mathcal{H}^d$ if $A = \bigcup_n A_n$ and $\forall n(\mathcal{H}^d(A_n) < \infty)$.

Theorem (Besicovitch-Davis)

If a Σ^1_1 set A is not σ-finite for $\mathcal{H}^d$, then A has a compact subset that is not σ-finite.

Algorithmic randomness descriptions of σ-finiteness (I)

Let

$$L_0(k) = \min\{n \mid \forall m \geq n K(m) \geq k\}$$

Theorem (Y.)

$\mathcal{H}^d(A) < \infty$ if and only if there is some c so that
$\exists x \exists^\infty k \forall z \in A \exists n > L_0^x(k)(K^x(z \upharpoonright n) < dn + k + c)$

So we obtain an algorithmic randomness description of σ-finiteness.

Algorithmic randomness descriptions of σ-finiteness (II)

Theorem (Lutz and Miller)

$\mathcal{H}^d(A) < \infty$ if and only if there is some c and real x so that $\forall z \in A \underline{\lim}_n KM^x(z \upharpoonright n) - dn < c$.

So we obtain an algorithmic randomness description of σ-finiteness.

A point-to-set proof of Besicovitch-Davis theorem

Slaman asked whether there is a point-to-se proof of Besicovitch-Davis theorem.
Miller: Borel determinacy+Lutz-Miller description gives a Borel version of Besicovitch-Davis theorem.
Y: Miller's result+Shoenfiled absoluteness gives the full version of Besicovitch-Davis theorem.

On Π^1_1-sets

For a real x, let ω_1^x be the least non-x-cursive ordinal.

Theorem (Spector, Gandy)

A set A is Π^1_1 if and only if there is a Σ_1 formula φ so that

$$x \in A \Leftrightarrow L_{\omega_1^x}[x] \models \varphi(x).$$

So a Π^1_1-set can be viewed as a recursively enumerable set.

On thin sets

A set A of reals is *thin* if A has no perfect subset.
For a real x, let ω_1^x be the least non-x-cursive ordinal.

Theorem (Mansfield, Solovay)

The set $\mathcal{C} = \{x \mid x \in L_{\omega_1^x}\}$ is the largest Π^1_1-thin set.

So every "master code" belongs to $\mathcal{C}$.

Slaman's result

Theorem (Slaman)

If $V = L$, then there is a thin Π^1_1 set A that is not σ-finite for $\mathcal{H}^d$.

Proof.

Slaman's original proof is a combination of two sophisticated methods from geometric measure theory and set theory.
Let $a_n = 2^{\sum_{m<n} a_m}$. Use the sequence partition ω. For any master code real x, there is some $y \equiv_T x'$ so that for any n,

- If $i \in [a_n, 2a_n - \log a_n)$, then $y(i) = 0$;
- If $i = a_n - \log a_n$, then $y(i) = x'(n)$;
- If $i \in [2a_n - \log a_n, a_{n+1})$, then $y(i) = \Omega^x(i)$.

Then the collection of such y's is the required set. □

谢谢