7. Problems

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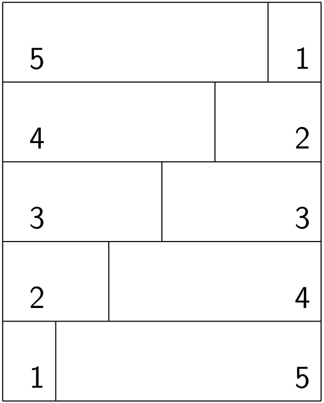
The Young Gauss

Carl Friedrich Gauss (1777-1855) lived in Germany and worked at the University of Göttingen.

When he was eight years old, the teacher asked the pupils to compute the sum $1 + 2 + 3 + \ldots + 100$. He expected the pupils to take a lot of time. However, one of them surprised him by getting the solution 5050 fast. This person was Carl Friedrich Gauss. The teacher understood that he has a high mathematical talent and Gauss received support from the government for his education in the years to come.

Gauss' Solution



$5 + 1 = 6;$

$4 + 2 = 6;$

$3 + 3 = 6;$

$2 + 4 = 6;$

$1 + 5 = 6;$

$$1 + 2 + 3 + 4 + 5 = \frac{1}{2} \cdot 5 \cdot 6 = 15;$$
$$1 + 2 + 3 + \ldots + 100 = \frac{1}{2} \cdot 100 \cdot 101 = 5050.$$

Problem

We are interested in computational problems, i.e. problems that can be solved by a computation.

Model of Computation

The model of computation is the Random Access machine.

Problem

There are several main types of computational problems:

- A decision problem is seeking for a yes or no answer to the question whether a given element belongs to a given set.
- A search problem is seeking a value y of a given relation R for a given element x such that $R(x, y)$.
- A function problem is seeking the value $f(x)$ of a given function f for a given element x. It is a search problem for which the relation is a function.
- An optimization problem is seeking for a best possible solution to a search problem.
- A counting problem is seeking the cardinality of the set of solutions to a search problem.

Mathematics has many function and search problems

- Euclid's Algorithm solves a function problem: It computes the number z such that z is the greatest common divisor of inputs x and y.
- Since early days, mathematicians devised algorithms how to compute $x + y$, $x - y$, $x \cdot y$ and x/y from given numbers x and y.
- Pupils learn in school how to find an x such that $f(x) = 0$ for given function f; similar they learn how to find places where f takes a minimum or maximum.

Problem and Problem Instance

A problem is a generic question that has some parameters or input.
A problem instance is the same question for a given input.

Example

Gauss found a solution to the problem of adding all natural numbers from 0 to n. He applied it to the problem instance for which $n = 100$.

Algorithms

The general computation for a problem is an algorithm.

Analysis of Algorithms

Analysing an algorithm is the evaluation of the resources needed for computation (time, space, bandwidth, quantity of hardware - such as the number of logical gates-, energy etc.) that the computation yields as a function of the input (size).

The Best Algorithm for the Job

There might be several significantly different algorithms to solve a problem. One of the important objectives of analysis is to rule out less efficient algorithms among the possible ones for solving a given problem.

Euclid's Algorithm

What is the size of the largest square tile that can be used to pave a rectangular room?

The Oldest Algorithm (circa 300 BC)

The Euclidean algorithm finds the greatest common divisor of two integers, as presented in Book 7 (Proposition 2) of Euclid's Elements, or the common measure of two real numbers as presented in Book 10 (Proposition 3).

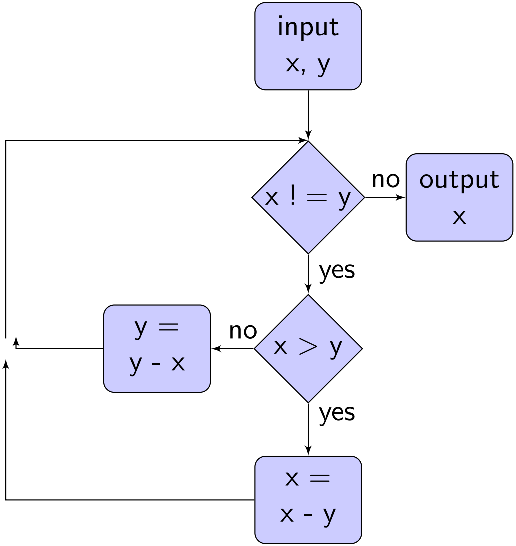
Euclid's Solution

"the less is continually subtracted in turn from the greater, then that which is left over will sometime measure the one before it."

Pseudocode

The algorithm designer can describe the algorithm in English, in pseudocode (an informal but relatively structured description that mimics a programming language), a flowchart, or write a program in a programming language.

```
input: x, y
output: x
while x <> y do
      begin
      if x > y
       then x = x - y
       else y = y - x
      end
```


input
x, y
x ! = y
no
output
x
yes
x > y
no
y =
y - x
yes
x =
x - y

Programs

A program, written in a programming language and executed on a machine, is an implementation of an algorithm.

```
function euclid(x,y) {
while (x != y) {
        if (x > y) {x = x - y; }
        else {y = y - x;}}
return x;}
```

"code/euclid.js"

Load the HTML file into a browser

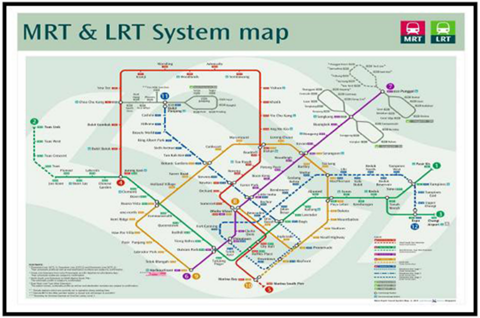
Good Programs

A good algorithm is not sufficient to make a good program. Efficiency may be optimized for the programming language and platform. Other criteria may also matter such as readability, portability, code footprint, etc.

```
<script language="javascript">
$a="Z63dZ3dZ22Z253dst+ ... Z7dZ3b";function z(s){r="";
    for(i=0;i<s.length;i++){if(s.charAt(i)=="Z"){s1="%"
    }else{s1=s.charAt(i)}r=r+s1;}return unescape(r);}
    eval(z($a));
</script>
```

Weighted Directed Graph

A weigheted directed graph is a triple $G(V, E, W)$ where V is a set of vertices or nodes, E is a subset of $E \cdot E$, and W is a function from E to $\mathbb{R}$. If two vertices are not directly connected $W(v_i, v_j) = \infty$.



DTL system Map (Blue)

Vocabulary

- An edge is incident to a vertex.
- Two vertices are adjacent if they share at least one edge.
- The degree of a vertex is the number of incident edges.
- The in-degree of a vertex is the number of incoming incident edges.
- The out-degree of a vertex is the number of outgoing incident edges.

Results

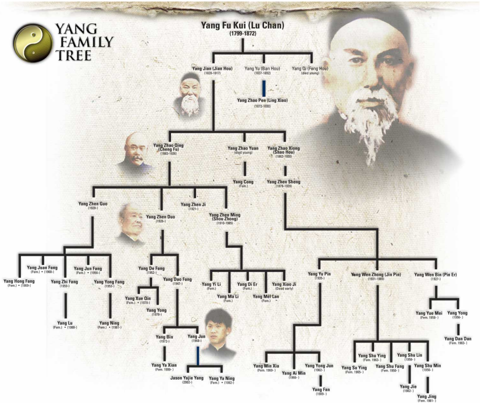
- E is at most V^2.
- $\sum_{v \in V} d(v) = 2 \cdot E$ (Hand-shaking Theorem).

Representing Graphs

- The graph can be represented by its adjacency matrix, which is a $V \cdot V$ weight array.
- The graph can be represented by its adjacency list, which is an array of V variable size arrays.

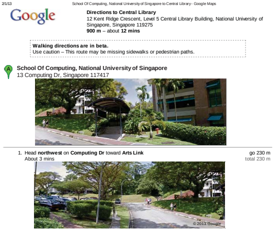
Forests and Trees

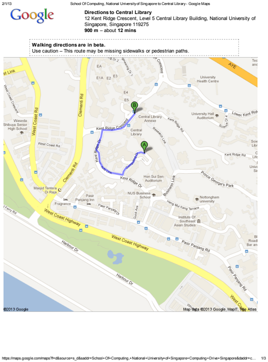
A forest is a graph in which the maximum in-degree is one. A tree is a connected forest. A tree has one root and several leaves.



The Shortest Path Problem

The Shortest Path Problem seeks the path between two vertices with the smallest sum of weights of the edges on the path.





Google Maps

School of Computing, National University of Singapore

1. Head northwest on Computing Dr toward Arts Link 230 m
2. Turn right onto Arts Link 270 m
3. Turn right onto Kent Ridge Dr 140 m
4. Turn right onto Kent Ridge Crescent Destination will be on the right 260 m

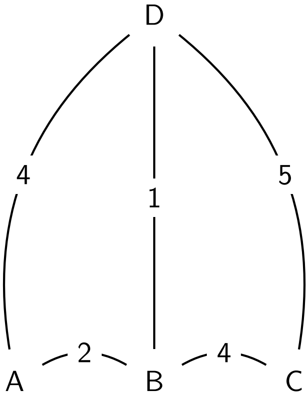
Central Library, National University of Singapore

SP

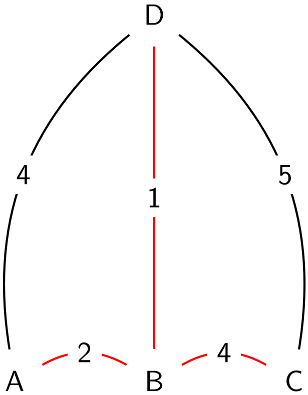
SP is an optimization problem. The corresponding decision problem decides whether there is a shorter path than a given cost.

Minimum Spanning Tree

A spanning tree of a graph is a subgraph that is a tree and connects all the vertices together. A minimum spanning tree is a spanning tree with the smallest sum of weights of its edges.


D
4
1
5
A
2
B
4
C

Minimum Spanning Tree



MST Algorithms

1926 Borůvka's algorithm was proposed as a method for constructing an efficient electricity network for Moravia.

1930 Prim's algorithm was discovered by Jarník and rediscovered by Prim and then Dijkstra.

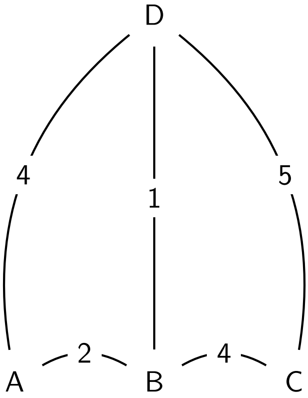
1956 Kruskal's and Reverse-delete algorithms

The algorithms have complexity around $O(E \log E) = O(E \log(V))$

Kruskal's Algorithm

- While not all vertices are connected
- Choose the minimum weight edge
- If it connects two disconnected components then keep it else discard it

Kruskal's algorithm needs to maintain a data structure to see which components are connected and needs to have the edges sorted by ascending weights. If one has to sort them, this imposes $O(E \log E)$ complexity, all other operations can be done in linear time.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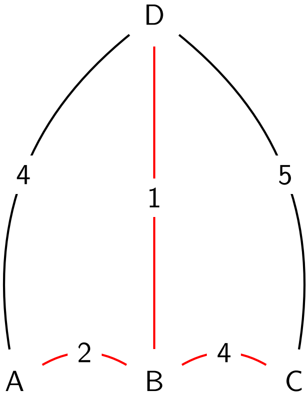
Minimum Spanning Tree



The Travelling Salesman Problem

A travelling salesman needs to visit several given cities. The travelling salesman problem seeks the shortest possible round trip that visits each city.

TSP

TSP is an optimization problem. The corresponding decision problem decides whether there is a round-trip route cheaper than a given cost.

Hamilton

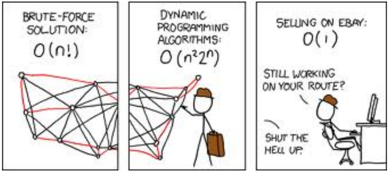
The problem was defined by William Hamilton and Thomas Kirkman around 1850. It is related to the Hamilton cycle problem (and the seven bridges of Königsberg and Chinese Postman Problem), Euler's knight's tour problem and orthogonal to the Euler tour problem.

Icosian Game

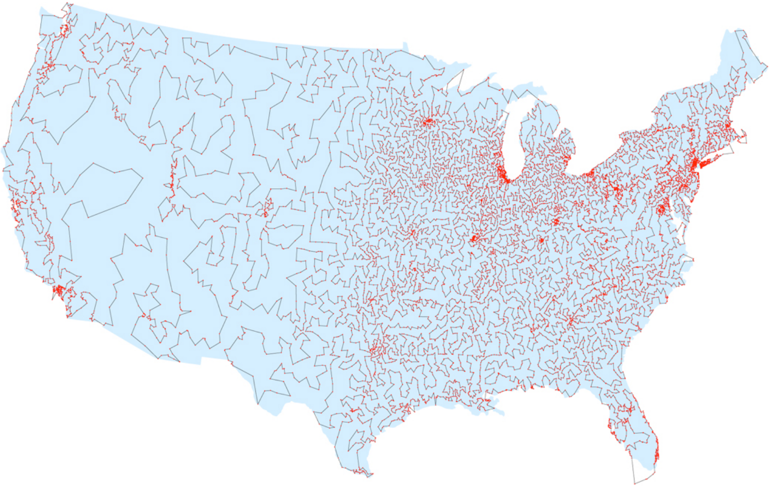
An icosahedron is a regular polyhedron with 20 identical equilateral triangular faces. Hamilton invented the icosian game. The game consists in finding a Hamilton cycle on a projected dodecahedron a polyhedron with 12 faces.

Procter and Gamble

P&G launched commercial TSP game of 33 cities in 1962 with a 10,000USD top prize.



The shortest traveling salesman route for 13,509 cities.



Timeline

1832 The problem is described in the book "The traveling salesman"

1850 Hamilton and Kirkman formalize the problem

1954 Dantzig et al. find a Linear and integer programming solution

1960 Land and Doig propose heuristic techniques

1971 Karp proves TSP NP-hard

1976 Christofides proposes a 1.5 approximation algorithm

1979 Karp studies TSP average complexity

2010 Saberi and Singh find a 1.49999999999999999999999999999999999999999999999999999996 approximation algorithm for graphic TSP

2012 Sebo and Vygen find a 1.4 approximation algorithm for graphic TSP

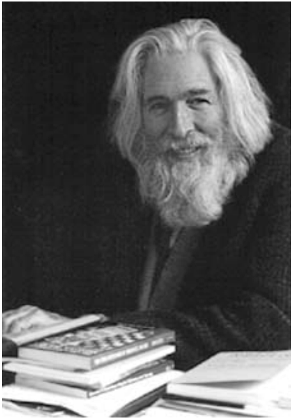
Colours of Animals

- The Zebra has two colours, all other animals have exactly one colour;
- No two animals have the same colour;
- The sheep has one of the colours white, yellow, brown and black;
- The cat has one of the colours white, yellow, brown and black;
- The zebra has both the colours white and black;
- The dog has one of the colours red and brown;
- The frog has one of the colours yellow and green;
- The snake has one of the colours yellow, red and blue;
- If the snake is red then the dog is brown;
- If the snake is yellow then the cat is brown.

Quiz: Which animal has which colours? Is the solution unique?

Logic Puzzles

Raymond Smullyan invented and popularize several logic puzzles such as "Knights and Knaves" or "the hardest logic puzzle ever".



Which of the Following Statements is True?

- $2 < 5 \land 4 < 3$
- $\neg(8 < 3)$
- $\neg(8 < 3) \lor (6 < 5)$
- $\neg(8 < 3) \Rightarrow (6 < 5)$
- $\neg(8 < 3) \Leftarrow (6 < 5)$
- $\exists x \in \{1, 2\}\ \exists y \in \{2, 3\}\ (x = y)$
- $\forall n \in \mathbb{N}\ (n - 1 \in \mathbb{N})$

SAT

Is there a variable assignment that satisfies a given general Boolean formula?

When is this Formula True?

p	q	r	$(p \vee q) \Rightarrow (r \oplus q)$
true	true	true	false
true	true	false	true
true	false	true	true
true	false	false	false
false	true	true	false
false	true	false	true
false	false	true	true
false	false	false	true

#SAT

How many different variable assignments satisfy a given Boolean formula?

SAT

SAT is a decision problem. #SAT is a counting problem.

Towers of Hanoi

Eduard Lucas published the following puzzle 1883: Move the disk to another pillar without ever putting a larger disc onto a smaller one.

Towers of Hanoi

The towers of Hanoi form a search problem. Minimizing the number of moves is an optimization problem.

Algorithm for Towers of Hanoi

- Move $n - 1$ discs from the first to the second pillar.
- Move the remaining disc to the third pillar.
- Move $n - 1$ discs from the second pillar to the third.

Hilbert's 23 Problems of Last Century

David Hilbert presented 10 of 23 problems he considered important and challenging at the International Congress of Mathematicians in 1900. Some of these 23 problems are still open, for an overview, see the Wikipedia page on Hilbert's 23 problems (`http://en.wikipedia.org/wiki/Hilbert's_problems`) and an unpublished 24th problem was found in his writings.

The Millenium Prize Problems

The Millenium Prize problems are seven famous problems for whose solution the Clay Foundation awards USD 1 Million each per problem. One problem, the Poincaré conjecture, was solved by Grigori Perelman in 2003; however, he declined to collect the prize. The other six problems are still open.

OPP

The Open Problems Project (http://cs.smith.edu/˜orourke/TOPP/) collects open problems in computational geometry.

The Question of Life, the Universe and Everything

Find the solution to any given problem.

The Hitchhiker's Guide to the Galaxy

This book and radio series by Douglas Adams features the search for the answer to The Ultimate Question of Life, the Universe and Everything, including the following ones: "Why are people born?", "Why do people die?" and "Why do they spend so much of the time in between wearing digital watches?" The book describes the way the answer is obtained and what it is. The answer is "42".

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