Puzzle of the Day:
A bag contains a collection of blue and red balls. Repeat:

• Take two balls from the bag.
• If they are the same color, discard them both and add a blue ball.
• If they are different colors, discard the blue ball and put the red ball back.

What do you know about the color of the final ball?
Summary

Last Week:

Toy example 1: array all 0’s?
• Gap-style question: All 0’s or far from all 0’s?

Toy example 2: Faction of 1’s?
• Additive $\pm \varepsilon$ approximation
• Hoeffding Bound

Is the graph connected?
• Gap-style question.
• $O(1)$ time algorithm.
• Correct with probability $2/3$.

Today:

Number of connected components in a graph.
• Additive approximation algorithm.

Weight of MST
• Multiplicative approximation algorithm.
Announcements / Reminders

Problem sets:

Problem Set 1 was due today.

Problem Set 2 will be released tonight.
Announcements / Reminders

Next Week: Guest Lecture

Arnab Bhattacharyya

Arnab’s research:
“My research area is theoretical computer science, in a broad sense. More specifically, I am interested in algorithms for big data, computational complexity, analysis and extremal combinatorics on finite fields, and algorithmic models for natural systems.”
Today’s Problem: Connected Components

Assumptions:

Graph $G = (V,E)$
- Undirected
- $n$ nodes
- $m$ edges
- maximum degree $d$

Error term: $\varepsilon$

Output:
Number of connected components.

Example: output 3
Today’s Problem: Connected Components

Approximation:

Output $C$ such that:

$$CC(G') - \varepsilon n \leq C \leq CC(G') + \varepsilon n$$

Alternate form:

$$|CC(G') - C| \leq \varepsilon n$$

Correct output: w.p. $> 2/3$

Example:

$\varepsilon = 1/10$

Output $\varepsilon \in \{2, 3, 4\}$
Today’s Problem: Connected Components

When is this useful?

What are trivial values of $\varepsilon$?

What are hard values of $\varepsilon$?

What sort of applications is this useful for?
Approximate Connected Components

When is this useful?

What are interesting values of $\varepsilon$?
- What happens when $\varepsilon = 1$?
- What happens when $\varepsilon = 1/(2n)$?

What sort of applications is this useful for?
- Large graphs?
- Large social networks?
- The internet?
- Networks with many connected components?
- Number of components follows a heavy tail distribution?
Approximate Connected Components

Key Idea 1:

Define: per-node cost

Let \( n(u) \) = number of nodes in the connected component containing node \( u \).
Define: per-node cost

Let $n(u)$ = number of nodes in the connected component containing node $u$.

Let $\text{cost}(u) = \frac{1}{n(u)}$. 
Approximate Connected Components

Key Idea 1:

Why is this useful?

\[ \sum_{u \in A} \text{cost}(u) = ?? \]

\[
\begin{align*}
\text{cost}(w) &= \frac{1}{6} \\
\text{cost}(x) &= \frac{1}{6} \\
\text{cost}(y) &= \frac{1}{3} \\
\text{cost}(z) &= 1
\end{align*}
\]
Why is this useful?

$$\sum_{u \in A} \text{cost}(u) = 1$$
Approximate Connected Components

Key Idea 1:

Why is this useful?

\[ \sum_{u \in A} \text{cost}(u) = 1 \]

\[ \sum_{u \in B} \text{cost}(u) = 1 \]

\[ \sum_{u \in C} \text{cost}(u) = 1 \]
Approximate Connected Components

Key Idea 1:

Why is this useful?

\[
\begin{align*}
\sum_{u \in A} \text{cost}(u) &= 1 \\
\sum_{u \in B} \text{cost}(u) &= 1 \\
\sum_{u \in C} \text{cost}(u) &= 1 \\
\sum_{u \in V} \text{cost}(u) &= \text{CC}(G)
\end{align*}
\]
Approximate Connected Components

Algorithm 1

\[
\sum_{u \in V} \text{cost}(u) = \text{CC}(G)
\]

\[
\text{sum} = 0
\]
for each \( u \) in \( V \):

\[
\text{sum} = \text{sum} + \text{cost}(u)
\]

return \( \text{sum} \)
Approximate Connected Components

Algorithm 1

\[
\text{sum} = 0 \\
\text{for each } u \in V: \\
\quad \text{sum} = \text{sum} + \text{cost}(u) \\
\text{return sum}
\]

Comments:
• Need a way to efficiently compute \( \text{cost}(u) \).
• Runs in \( O(n) \) time.
Choose a small random subset $S$ of $V$. For each node $u$ in $S$, compute $\text{cost}(u)$. Use the sample to estimate the average cost of all the nodes.

Sample

- Choose a small random subset $S$ of $V$.
- For each node $u$ in $S$, compute $\text{cost}(u)$.
- Use the sample to estimate the average cost of all the nodes.
Approximate Connected Components

Key Idea 2: Sampling

Worries?

\[
cost(w) = \frac{1}{6} \quad cost(x) = \frac{1}{6} \quad cost(y) = \frac{1}{3} \quad cost(z) = 1
\]
Approximate Connected Components

Key Idea 2: Sampling

Worries?

• Big components are sampled more often than small components?
• Small components may never be sampled?
• Bad examples?
  1 component of size 90,
  10 components of size 1
sum = 0
for j = 1 to s:
    Choose u uniformly at random.
    sum = sum + cost(u)
return n·(sum/s)

Comments:
• (sum/s) is average cost of sample.
• Efficiently compute cost(u)?
• Runs in O(s) time.
Approximate Connected Components

Algorithm 2 Analysis

Define random variables: \( Y_1, Y_2, \ldots, Y_s \)

\[
\begin{align*}
    u_j &= \text{node chosen in } j^\text{th} \text{ iteration} \\
    Y_j &= \text{cost}(u_j)
\end{align*}
\]
Algorithm 2 Analysis

\[ Y_j = \text{cost}(u_j) \]

sum = 0
for j = 1 to s:
    Choose u uniformly at random.
    sum = sum + cost(u)
return n\cdot (sum/s)

\[
E[Y_j] = \sum_{i=1}^{n} \frac{1}{n} \text{cost}(u_i)
\]
Approximate Connected Components

Algorithm 2 Analysis

\[ Y_j = \text{cost}(u_j) \]

\[ E[Y_j] = \sum_{i=1}^{n} \frac{1}{n} \text{cost}(u_i) = \frac{1}{n} \sum_{i=1}^{n} \text{cost}(u_i) \]

sum = 0
for \( j = 1 \) to \( s \):
    Choose \( u \) uniformly at random.
    \( \text{sum} = \text{sum} + \text{cost}(u) \)
return \( n \cdot (\text{sum}/s) \)
Approximate Connected Components

Algorithm 2 Analysis

\[ Y_j = \text{cost}(u_j) \]

\[
E [Y_j] = \sum_{i=1}^{n} \frac{1}{n} \text{cost}(u_i) = \frac{1}{n} \sum_{i=1}^{n} \text{cost}(u_i)
\]

\[ = \frac{1}{n} \text{CC}(G) \]
Approximate Connected Components

Algorithm 2 Analysis

\[
\text{sum} = 0 \\
\text{for } j = 1 \text{ to } s: \\
\quad \text{Choose } u \text{ uniformly at random.} \\
\quad \text{sum} = \text{sum} + \text{cost}(u) \\
\text{return } n \cdot (\text{sum}/s)
\]

\[
Y_j = \text{cost}(u_j) \\
E [Y_j] = \frac{1}{n} \text{CC}(G)
\]

\[
E \left[ \sum_{j=1}^{s} Y_j \right] = s E [Y_j] = \frac{s}{n} \text{CC}(G)
\]
Algorithm 2 Analysis

\begin{align*}
\text{sum} &= 0 \\
\text{for } j = 1 \text{ to } s: & \quad \text{Choose } u \text{ uniformly at random.} \\
\text{sum} &= \text{sum} + \text{cost}(u) \\
\text{return } n \cdot (\text{sum}/s) \\

Y_j &= \text{cost}(u_j) \\
E[Y_j] &= \frac{1}{n} \text{CC}(G) \\
E\left[\sum_{j=1}^{s} Y_j\right] &= \frac{s}{n} \text{CC}(G)
\end{align*}
Approximate Connected Components

Algorithm 2 Analysis

```
sum = 0
for j = 1 to s:
    Choose u uniformly at random.
    sum = sum + cost(u)
return n · (sum/s)
```

Notice:
Output of algorithm is:

\[
\frac{n}{s} \sum_{j=1}^{s} Y_j
\]

\[
Y_j = \text{cost}(u_j)
\]

\[
E[Y_j] = \frac{1}{n} \cdot \text{CC}(G)
\]

\[
E \left[ \sum_{j=1}^{s} Y_j \right] = \frac{s}{n} \cdot \text{CC}(G)
\]
Approximate Connected Components

Algorithm 2 Analysis

sum = 0
for j = 1 to s:
    Choose u uniformly at random.
    sum = sum + cost(u)
return n \cdot (sum/s)

Notice:

Expected output of algorithm is:

\[ E \left[ n \cdot \left( \frac{\text{sum}}{s} \right) \right] = \frac{n}{s} \left( \frac{s}{n} \text{CC}(G) \right) = \text{CC}(G) \]
Approximate Connected Components

Algorithm 2 Analysis

sum = 0
for j = 1 to s:
    Choose u uniformly at random.
    sum = sum + cost(u)
return n·(sum/s)

Important step:
Expected out is number of connected components!

(Algorithm is an unbiased estimator.)
Approximate Connected Components

Algorithm 2 Analysis

\[
\text{sum} = 0 \\
\text{for } j = 1 \text{ to } s:\ \\
\quad \text{Choose } u \text{ uniformly at random.} \\
\quad \text{sum} = \text{sum} + \text{cost}(u) \\
\text{return } n \cdot (\text{sum}/s)
\]

Notice:

Goal:
\[
\Pr \left\{ \left| \text{CC}(G) - \frac{n}{s} \sum_{j=1}^{s} Y_j \right| > \epsilon n \right\} \leq 1/3
\]
Approximate Connected Components

Algorithm 2 Analysis

```plaintext
sum = 0
for j = 1 to s:
    Choose u uniformly at random.
    sum = sum + cost(u)
return n \cdot (sum/s)
```

Notice:

Goal:

\[
\Pr \left\{ \left| \frac{CC(G)}{s} - \frac{n}{s} \sum_{j=1}^{s} Y_j \right| > \frac{\epsilon n}{2} \right\} \leq \frac{1}{3}
\]
Given: independent random variables $Y_1, Y_2, ..., Y_s$
Assume: each $Y_j \in [0,1]$

Then:

$$\Pr \left\{ \left| \mathbb{E} \left[ \sum_{j=1}^{s} Y_j \right] - \sum_{j=1}^{s} Y_j \right| > t \right\} \leq 2e^{-2t^2/s}$$
Given: independent random variables $Y_1, Y_2, ..., Y_s$
Assume: each $Y_j \in [0,1]$
Then:

$$\Pr \left\{ \left| E \left[ \sum_{j=1}^{s} Y_j \right] - \sum_{j=1}^{s} Y_j \right| > t \right\} \leq 2e^{-2t^2/s}$$

Goal:

$$\Pr \left\{ \left| CC(G) - \frac{n}{s} \sum_{j=1}^{s} Y_j \right| > \epsilon n/2 \right\} \leq 1/3$$
Approximate Connected Components

Algorithm 2 Analysis

Derivation:

\[ \Pr \left\{ \left| CC(G) - \frac{n}{s} \sum_{j=1}^{s} Y_j \right| > \epsilon n/2 \right\} = \]
Approximate Connected Components

Algorithm 2 Analysis

Derivation:

\[
Pr \left\{ \left| \frac{n}{s} \sum_{j=1}^{s} Y_j - \frac{n}{n} \right| > \epsilon n/2 \right\} = Pr \left\{ \left| E \left[ \frac{n}{s} \sum_{i=1}^{s} Y_i \right] - \frac{n}{s} \sum_{j=1}^{s} Y_j \right| > \epsilon n/2 \right\}
\]

\[
E \left[ \sum_{j=1}^{s} Y_j \right] = \frac{s}{n} \text{CC}(G)
\]
Approximate Connected Components

Algorithm 2 Analysis

Derivation:

\[ \Pr \left\{ \left| \text{CC}(G) - \frac{n}{s} \sum_{j=1}^{s} Y_j \right| > \epsilon n/2 \right\} = \Pr \left\{ \left| \mathbb{E} \left[ \frac{n}{s} \sum_{i=1}^{s} Y_i \right] - \frac{n}{s} \sum_{j=1}^{s} Y_j \right| > \epsilon n/2 \right\} \]

\[ = \Pr \left\{ \left| \mathbb{E} \left[ \sum_{i=1}^{s} Y_i \right] - \sum_{i=1}^{s} Y_j \right| > \frac{s}{n} \epsilon n/2 \right\} \]
Approximate Connected Components

Algorithm 2 Analysis

Derivation:

\[
\Pr \left\{ \left| CC(G) - \frac{n}{s} \sum_{j=1}^{s} Y_j \right| > \epsilon n/2 \right\} = \Pr \left\{ \left| \mathbb{E} \left[ \frac{n}{s} \sum_{i=1}^{s} Y_i \right] - \frac{n}{s} \sum_{j=1}^{s} Y_j \right| > \epsilon n/2 \right\}
\]

\[
= \Pr \left\{ \left| \mathbb{E} \left[ \sum_{i=1}^{s} Y_i \right] - \sum_{j=1}^{s} Y_j \right| > \frac{s}{n} \epsilon n/2 \right\}
\]

\[
= \Pr \left\{ \left| \mathbb{E} \left[ \sum_{i=1}^{s} Y_i \right] - \sum_{j=1}^{s} Y_j \right| > \epsilon s/2 \right\}
\]
Approximate Connected Components

\[
\Pr \left\{ \left| \mathbb{E} \left( \sum_{j=1}^{s} Y_j \right) - \sum_{j=1}^{s} Y_j \right| > t \right\} \leq 2e^{-2t^2/s}
\]

\[
\Pr \left\{ \left| \text{CC}(G) - \frac{n}{s} \sum_{j=1}^{s} Y_j \right| > \epsilon n/2 \right\} = \Pr \left\{ \left| \mathbb{E} \left( \frac{n}{s} \sum_{i=1}^{s} Y_i \right) - \frac{n}{s} \sum_{j=1}^{s} Y_j \right| > \epsilon n/2 \right\}
\]

\[
= \Pr \left\{ \left| \mathbb{E} \left( \sum_{i=1}^{s} Y_i \right) - \sum_{j=1}^{s} Y_j \right| > \frac{s}{n} \epsilon n/2 \right\}
\]

\[
= \Pr \left\{ \left| \mathbb{E} \left( \sum_{i=1}^{s} Y_i \right) - \sum_{j=1}^{s} Y_j \right| > \epsilon s/2 \right\}
\]
Approximate Connected Components

Algorithm 2 Analysis

Derivation:

\[
\Pr \left\{ \left| \text{CC}(G) - \frac{n}{s} \sum_{j=1}^{s} Y_j \right| > \epsilon n/2 \right\} =
\]

\[
\Pr \left\{ \left| \mathbb{E} \left[ \sum_{i=1}^{s} Y_i \right] - \sum_{j=1}^{s} Y_j \right| > \epsilon s/2 \right\} \leq 2e^{-2(\epsilon s/2)^2/s}
\]
Approximate Connected Components

Algorithm 2 Analysis

Derivation:

$$\Pr \left\{ \left| \text{CC}(G) - \frac{n}{s} \sum_{j=1}^{s} Y_j \right| > \epsilon n/2 \right\} =$$

$$\Pr \left\{ \left| \mathbb{E} \left[ \sum_{i=1}^{s} Y_i \right] - \sum_{j=1}^{s} Y_j \right| > \epsilon s/2 \right\} \leq 2e^{-2(\epsilon s/2)^2 / s}$$

$$\leq 2e^{-2\epsilon^2 s/4}$$
Approximate Connected Components

Algorithm 2 Analysis

Derivation:

\[
\Pr \left\{ \left| CC(G) - \frac{n}{s} \sum_{j=1}^{s} Y_j \right| > \epsilon n/2 \right\} =
\]

\[
\Pr \left\{ \left| \mathbb{E} \left[ \sum_{i=1}^{s} Y_i \right] - \sum_{j=1}^{s} Y_j \right| > \epsilon s/2 \right\} \leq 2e^{-2(\epsilon s/2)^2/s}
\]

\[
S = \frac{4}{\epsilon^2}
\]
Approximate Connected Components

Algorithm 2 Analysis

Derivation:

\[
\Pr \left\{ \left| \text{CC}(G) - \frac{n}{s} \sum_{j=1}^{s} Y_j \right| > \epsilon n/2 \right\} =
\]

\[
\Pr \left\{ \left| \mathbb{E} \left[ \sum_{i=1}^{s} Y_i \right] - \sum_{j=1}^{s} Y_j \right| > \epsilon s/2 \right\} \leq 2e^{-2(\epsilon s/2)^2 / s}
\]

\[
\leq 2e^{-2\epsilon^2 s/4}
\]

\[
\leq 2e^{-\epsilon^2 (4/\epsilon^2) / 2}
\]

\[
S = \frac{4}{\epsilon^2}
\]
Approximate Connected Components

Algorithm 2 Analysis

Derivation:

\[
\Pr \left\{ \left| \text{CC}(G) - \frac{n}{s} \sum_{j=1}^{s} Y_j \right| > \epsilon n/2 \right\} =
\]

\[
\Pr \left\{ \left| \mathbb{E} \left[ \sum_{i=1}^{s} Y_i \right] - \sum_{j=1}^{s} Y_j \right| > \epsilon s/2 \right\} \leq 2e^{-2(\epsilon s/2)^2/s}
\]

\[
S = \frac{4}{\epsilon^2}
\]

\[
\leq 2e^{-2\epsilon^2 s/4}
\]

\[
\leq 2e^{\epsilon^2(4/\epsilon^2)/2}
\]

\[
\leq 2e^{-2}
\]

\[
< 1/3
\]
Approximate Connected Components

Algorithm 2

\[
\text{sum} = 0 \\
\text{for } j = 1 \text{ to } s: \\
\quad \text{Choose } u \text{ uniformly at random.} \\
\quad \text{sum} = \text{sum} + \text{cost}(u) \\
\text{return } n \cdot (\text{sum}/s)
\]

We have shown:

W.p. > 2/3, output is equal to:

\[\text{CC}(G) \pm \epsilon n/2\]
Approximate Connected Components

Algorithm 2

sum = 0
for j = 1 to s:
    Choose u uniformly at random.
    sum = sum + cost(u)
return n \cdot \frac{\text{sum}}{s}

We have shown:
Time: $O(1/\varepsilon^2)$
Key problem:
How to efficiently compute\[\text{cost}(u)\].

Approximate Connected Components

Key Idea 2: Sampling

\[\begin{align*}
cost(w) &= \frac{1}{6} \\
cost(x) &= \frac{1}{6} \\
cost(y) &= \frac{1}{3} \\
cost(z) &= 1
\end{align*}\]
Approximate Connected Components

Key Idea 2: Sampling

Key problem:
How to efficiently compute $\text{cost}(u)$.

Key idea 3:
Approximate $\text{cost}(u)$. 

A

B

C

$\text{cost}(w) = 1/6$

$\text{cost}(x) = 1/6$

$\text{cost}(y) = 1/3$

$\text{cost}(z) = 1$
Approximate Connected Components

Key Idea 3: Approximate Cost

Approximate low cost components:
If \( \text{cost}(u) \) is small, round up.

How small is small enough?
Approximate Connected Components

Key Idea 3: Approximate Cost

Approximate low cost components:
If $\text{cost}(u) < \varepsilon/2$, round up.
Approximate Connected Components

Key Idea 3: Approximate Cost

Ignore low cost components:
If \( \text{cost}(u) < \frac{\varepsilon}{2} \), round up.

Total added cost \( \leq \frac{\varepsilon n}{2} \).
Approximate Connected Components

Key Idea 3: Approximate Cost

Define: per-node cost

Let $n(u)$ = number of nodes in the connected component containing node $u$.

Let $\bar{n}(u) = \min(n(u), 2/\varepsilon)$.

Let $\text{cost}(u) = \max(1/n(u), \varepsilon/2) = 1/\bar{n}(u)$.
Approximate Connected Components

Key Idea 3: Approximate Cost

Define: per-node cost

Let \( n(u) \) = number of nodes in the connected component containing node \( u \).

Let \( \tilde{n}(u) = \min(n(u), 2/\varepsilon) \).

Let \( \text{cost}(u) = \max(1/n(u), \varepsilon/2) \).

\[ = 1/\tilde{n}(u). \]

Define:

\[ \overline{C} = \sum_{u \in V} \text{cost}(u) \]

Note:

\[ n(u) \geq \overline{n}(u) \]
\[ 1/n(u) \leq 1/\overline{n}(u) \]
Approximate Connected Components

Key Idea 3: Approximate Cost

Define: per-node cost

Let $n(u)$ = number of nodes in the connected component containing node $u$.

Let $\tilde{n}(u) = \min(n(u), 2/\varepsilon)$.

Let $\text{cost}(u) = \max(1/n(u), \varepsilon/2)$.

Define:

$$\bar{C} = \sum_{u \in V} \text{cost}(u)$$

Note:

$$n(u) \geq \bar{n}(u)$$
$$1/n(u) \leq 1/\bar{n}(u)$$
Approximate Connected Components

Close enough approximation:

\[ |CC(G) - \bar{C}| = \bar{C} - CC(G) \]

\[
\begin{align*}
n(u) & \geq \bar{n}(u) \\
1/n(u) & \leq 1/\bar{n}(u)
\end{align*}
\]

Intuition:
By rounding cost(u) up to \( \varepsilon /2 \), we increase the error at most \( \varepsilon n/2 \).
Approximate Connected Components

Close enough approximation:

\[ |CC(G) - \bar{C}| = \bar{C} - CC(G) \]

\[ = \sum_{j=1}^{n} 1/\bar{n}(u) - \sum_{j=1}^{n} 1/n(u) \]

**Intuition:**
By rounding \( \text{cost}(u) \) up to \( \varepsilon/2 \), we increase the error at most \( \varepsilon n/2 \).
Approximate Connected Components

Close enough approximation:

\[ |CC(G) - \bar{C}| = \bar{C} - CC(G) \]

\[ = \sum_{j=1}^{n} 1/\bar{n}(u) - \sum_{j=1}^{n} 1/n(u) \]

\[ = \sum_{j=1}^{n} (1/\bar{n}(j) - 1/n(j)) \]

**Intuition:**
By rounding cost(u) up to \( \varepsilon/2 \), we increase the error at most \( \varepsilon n/2 \).
Approximate Connected Components

Close enough approximation:

$$\left| \text{CC}(G) - \overline{C} \right| = \overline{C} - \text{CC}(G)$$

$$= \sum_{j=1}^{n} 1/\overline{n}(u) - \sum_{j=1}^{n} 1/n(u)$$

$$= \sum_{j=1}^{n} (1/\overline{n}(j) - 1/n(j))$$

$$\leq \sum_{j=1}^{n} 1/\overline{n}(j)$$

**Intuition:**
By rounding cost(u) up to $\varepsilon/2$, we increase the error at most $\varepsilon n/2$. 
Approximate Connected Components

Close enough approximation:

\[
|CC(G) - \overline{C}| = \overline{C} - CC(G)
\]

\[
= \sum_{j=1}^{n} 1/\overline{n}(u) - \sum_{j=1}^{n} 1/n(u)
\]

\[
= \sum_{j=1}^{n} (1/\overline{n}(j) - 1/n(j))
\]

\[
\leq \sum_{j=1}^{n} 1/\overline{n}(j)
\]

\[
\leq \sum_{j=1}^{n} \epsilon/2
\]

**Intuition:**
By rounding cost(u) up to \(\epsilon/2\), we increase the error at most \(\epsilon n/2\).
Approximate Connected Components

Close enough approximation:

\[ |CC(G) - \overline{C}| = \overline{C} - CC(G) \]

\[ = \sum_{j=1}^{n} \frac{1}{\overline{n}(u)} - \sum_{j=1}^{n} \frac{1}{n(u)} \]

\[ = \sum_{j=1}^{n} \left( \frac{1}{\overline{n}(j)} - \frac{1}{n(j)} \right) \]

\[ \leq \sum_{j=1}^{n} \frac{1}{\overline{n}(j)} \]

\[ \leq \sum_{j=1}^{n} \frac{\epsilon}{2} \]

\[ \leq \epsilon n / 2 \]

**Intuition:**
By rounding cost(u) up to \( \epsilon/2 \), we increase the error at most \( \epsilon n/2 \).
Approximate Connected Components

Algorithm 3

\[
\text{sum} = 0 \\
\text{for } j = 1 \text{ to } s: \\
\quad \text{Choose } u \text{ uniformly at random.} \\
\quad \text{sum} = \text{sum} + \text{cost}(u) \\
\text{return } n \cdot (\text{sum}/s)
\]

We have shown:
Sufficient to approximate \( \text{cost}(u) \) by rounding up.

Costs of vertices:
- \( \text{cost}(w) = 1/6 \)
- \( \text{cost}(x) = 1/6 \)
- \( \text{cost}(y) = 1/3 \)
- \( \text{cost}(z) = 1 \)
Approximate Connected Components

Algorithm 3

Define: per-node cost

Let $n(u) =$ number of nodes in the connected component containing node $u$.

Let $\tilde{n}(u) = \min(n(u), 2/\varepsilon)$.

Let $\text{cost}(u) = \max(1/n(u), \varepsilon/2)$.
  \[ = 1/\tilde{n}(u). \]

How to efficiently compute $\text{cost}(u)$?
Approximate Connected Components

Algorithm 3

Define: per-node cost

Let $n(u)$ = number of nodes in the connected component containing node $u$.

Let $\tilde{n}(u) = \min(n(u), 2/\varepsilon)$.

Let $\text{cost}(u) = \max(1/n(u), \varepsilon/2)$.

$= 1/\tilde{n}(u)$.

How to efficiently compute $\text{cost}(u)$?
sum = 0
for j = 1 to s:
    Choose \( u \) uniformly at random.
    Perform a BFS from \( u \); stop after seeing \( 2/\varepsilon \) nodes.
if BFS found > \( 2/\varepsilon \) nodes:
    sum = sum + \( \varepsilon/2 \)
else if BFS found \( n(u) \) nodes:
    sum = sum + \( 1/n(u) \)
return \( n \cdot (\text{sum}/s) \)
Approximate Connected Components

Analysis

Goal:
\[ \left| \frac{n}{s} \cdot \text{sum} - \bar{C} \right| \leq \epsilon n / 2 \]
Approximate Connected Components

Analysis

Goal:
\[ \left| \frac{n}{s} \cdot \text{sum} - \overline{C} \right| \leq \frac{\epsilon n}{2} \]

Implies:
\[
\left| \frac{n}{s} \cdot \text{sum} - \text{CC}(G) \right| \leq \left| \frac{n}{s} \cdot \text{sum} - \overline{C} \right| + \left| \overline{C} - \text{CC}(G) \right|
\leq \frac{\epsilon n}{2} + \frac{\epsilon n}{2}
\leq \epsilon n
\]
Approximate Connected Components

Algorithm 3 Analysis

Define random variables: $Y_1, Y_2, ..., Y_s$

$u_j = \text{node chosen in } j^{th} \text{ iteration}$

$Y_j = \text{cost}(u_j)$

Rounded up cost
Define random variables: $Y_1, Y_2, ..., Y_s$

$$E[Y_j] = \sum_{i=1}^{n} \frac{1}{n} \text{cost}(u_i) = \frac{1}{n} \sum_{i=1}^{n} \text{cost}(u_i)$$

$$= \frac{1}{n} C$$
Approximate Connected Components

Algorithm 3 Analysis

Unbiased estimator:

\[
E \left[ \sum_{j=1}^{s} Y_j \right] = sE \left[ Y_j \right] = \frac{s}{\bar{C} n}
\]
Expected output of algorithm is:

\[
E \left[ n \cdot \left( \frac{\text{sum}}{s} \right) \right] = \frac{n}{s} \left( \frac{s}{n} \frac{\bar{C}}{} \right) = \bar{C}
\]
Approximate Connected Components

Algorithm 3 Analysis

Goal:

\[
Pr \left\{ \left| \overline{C} - \frac{n}{s} \sum_{j=1}^{s} Y_j \right| > \epsilon n / 2 \right\} \leq 1/3
\]
Approximate Connected Components

Algorithm 3 Analysis

Derivation:

\[
\Pr \left\{ \left| \overline{C} - \frac{n}{s} \sum_{j=1}^{s} Y_j \right| > \epsilon n/2 \right\} = \Pr \left\{ \left| \mathbb{E} \left[ \frac{n}{s} \sum_{i=1}^{s} Y_i \right] - \frac{n}{s} \sum_{j=1}^{s} Y_j \right| > \epsilon n/2 \right\}
\]

\[
= \Pr \left\{ \left| \mathbb{E} \left[ \sum_{i=1}^{s} Y_i \right] - \sum_{j=1}^{s} Y_j \right| > \frac{s \epsilon n}{2} \right\}
\]

\[
= \Pr \left\{ \left| \mathbb{E} \left[ \sum_{i=1}^{s} Y_i \right] - \sum_{j=1}^{s} Y_j \right| > \epsilon s/2 \right\}
\]
Approximate Connected Components

Algorithm 3 Analysis

Derivation:

\[
\Pr \left\{ \left\| \widetilde{C} - \frac{n}{s} \sum_{j=1}^{s} Y_j \right\| > \epsilon n / 2 \right\} = \\
\Pr \left\{ \left\| \mathbb{E} \left[ \sum_{i=1}^{s} Y_i \right] - \sum_{j=1}^{s} Y_j \right\| > \epsilon s / 2 \right\} \leq 2e^{-2(\epsilon s/2)^2/s} \\
\leq 2e^{-2\epsilon^2 s / 4} \\
\leq 2e^{-\epsilon^2 (4/\epsilon^2 s)^2 / 2} \\
\leq 2e^{-2} \\
< 1/3
\]

\[
S = \frac{4}{\epsilon^2}
\]
Approximate Connected Components

Analysis

Goal:
\[ \left| \frac{n}{s} \cdot \text{sum} - \bar{C} \right| \leq \frac{\epsilon n}{2} \]

Implies:
\[
\left| \frac{n}{s} \cdot \text{sum} - \text{CC}(G) \right| \leq \left| \frac{n}{s} \cdot \text{sum} - \bar{C} \right| + \left| \bar{C} - \text{CC}(G) \right|
\leq \frac{\epsilon n}{2} + \frac{\epsilon n}{2}
\leq \epsilon n
\]
Approximate Connected Components

Algorithm 3

sum = 0
for j = 1 to s:
    Choose u uniformly at random.
    Perform a BFS from u; stop after seeing $2/\varepsilon$ nodes.
    if BFS found $> 2/\varepsilon$ nodes:
        sum = sum + $\varepsilon/2$
    else if BFS found $n(u)$ nodes:
        sum = sum + $1/n(u)$
return $n \cdot (\text{sum}/s)$
We have shown:

With probability $> \frac{2}{3}$, output is equal to: $\text{CC}(G) \pm \varepsilon n$
Approximate Connected Components

Algorithm 3

\[
\text{sum} = 0
\]

for \( j = 1 \) to \( s \):

Choose \( u \) uniformly at random.

Perform a BFS from \( u \); stop after seeing \( \frac{2}{\varepsilon} \) nodes.

if BFS found > \( \frac{2}{\varepsilon} \) nodes:

\[
\text{sum} = \text{sum} + \frac{\varepsilon}{2}
\]

else if BFS found \( n(u) \) nodes:

\[
\text{sum} = \text{sum} + \frac{1}{n(u)}
\]

return \( n \cdot \left( \frac{\text{sum}}{s} \right) \)

Cost of BFS: \( O(\frac{2}{\varepsilon} \cdot d) \)
Approximate Connected Components

Algorithm 3

```
sum = 0
for j = 1 to s:
    Choose u uniformly at random.
    Perform a BFS from u; stop after seeing $2/\varepsilon$ nodes.
    if BFS found > $2/\varepsilon$ nodes:
        sum = sum + $\varepsilon/2$
    else if BFS found $n(u)$ nodes:
        sum = sum + $1/n(u)$
return $n \cdot (\text{sum}/s)$
```

Cost of BFS: $O((2 / \varepsilon) \cdot d)$

Total cost:

$O(s(2/\varepsilon) \cdot d) = O((1/\varepsilon^2)(2/\varepsilon)d) = O(d/\varepsilon^3)$
We have shown:

With probability $> \frac{2}{3}$, output is equal to: $\text{CC}(G) \pm \varepsilon n$

Running time: $O\left(\frac{d}{\varepsilon^3}\right)$
We have shown:

With probability $> 1 - \frac{1}{\delta}$, output is equal to:

$$\text{CC}(G) \pm \varepsilon n$$

Running time: $O\left(\frac{d \ln \delta}{\varepsilon^3}\right)$
Summary

Last Week:

Toy example 1: array all 0’s?
- Gap-style question: All 0’s or far from all 0’s?

Toy example 2: Faction of 1’s?
- Additive $\pm \varepsilon$ approximation
- Hoeffding Bound

Is the graph connected?
- Gap-style question.
- $O(1)$ time algorithm.
- Correct with probability $2/3.$

Today:

Number of connected components in a graph.
- Approximation algorithm.

Weight of MST
- Approximation algorithm.

9 dots
4 lines
Today’s Problem: Minimum Spanning Tree

Assumptions:

Graph G = (V,E)
- Undirected
- Weighted, max weight W
- Connected
- n nodes
- m edges
- maximum degree d

Error term: ε < 1/2

Output:
Weight of MST.

Example: output 16
Approximation:

Output $M$ such that:

$\text{MST}(G)(1 - \epsilon) \leq M \leq \text{MST}(1 + \epsilon)$

Alternate form:

$M = \text{MST}(G)(1 \pm \epsilon)$

Correct output: w.p. > $2/3$

Example:

$\epsilon = 1/4$

Output $\in [12, 20]$
Today’s Problem: Minimum Spanning Tree

When is this useful?

What are trivial values of $\varepsilon$?

What are hard values of $\varepsilon$?

What sort of applications is this useful for?

Why multiplicative approximation for MST and additive approximation for connected components?
Simple Minimum Spanning Tree

Assume all weights 1 or 2

Which edges must be in MST?

How many weight-2 edges in MST?

Best (exact) algorithm?
Simple Minimum Spanning Tree

Assume all weights 1 or 2

Let $G_1 = \text{graph containing only edges of weight 1.}$
Simple Minimum Spanning Tree

Assume all weights 1 or 2

Let $G_1$ = graph containing only edges of weight 1.

Let $C_1$ = number of connected components in $G_1$.

Ex: $C_1 = 6$
Simple Minimum Spanning Tree

Assume all weights 1 or 2

Let $G_1$ = graph containing only edges of weight 1.

Let $C_1$ = number of connected components in $G_1$.

Claim: MST contains example $C_1 - 1$ edges of weight 2.

Ex: $C_1 = 6$
Claim: MST contains example $C_1 - 1$ edges of weight 2.

Basic MST Property:
For any cut, minimum weight edge across cut is in MST.

Ex: $C_1 = 6$
Simple Minimum Spanning Tree

Claim: MST contains example $C_1 - 1$ edges of weight 2.

Algorithm:

For any connected component, add minimum weight outgoing edge.

Here all the edges have weight 2, so add $C_1 - 1$ edges of weight 2.

Ex: $C_1 = 6$
Simple Minimum Spanning Tree

Claim: MST contains example $C_1 - 1$ edges of weight 2.

Weight of MST?

Assume all weights 1 or 2

Ex: $C_1 = 6$
Simple Minimum Spanning Tree

Assume all weights 1 or 2

Claim: MST contains example $C_1 - 1$ edges of weight 2.

Weight of MST?

\[ (n - (C_1 - 1) - 1) \cdot 1 + (C_1 - 1) \cdot 2 \]

\[ = n + C_1 - 2 \]

Ex: $10 + 6 - 2 = 14$

Ex: $C_1 = 6$
Simple Minimum Spanning Tree

Assume all weights 1 or 2

Weight of MST: \( n + C_1 - 2 \)

Algorithm idea?

Ex: \( C_1 = 6 \)
Simple Minimum Spanning Tree

Assume all weights 1 or 2

Weight of MST: \( n + C_1 - 2 \)

Algorithm idea:
Approximate connected components of \( G_1 \).

Ex: \( C_1 = 6 \)
Approximate Minimum Spanning Tree

Weights \( \{1, 2, \ldots, W\} \)

Let \( G_1 \) = graph containing only edges of weight 1.

Let \( G_2 \) = graph containing only edges of weight \( \{1, 2\} \).

\[ \ldots \]

Let \( G_j \) = graph containing only edges of weights \( \{1, 2, \ldots, j\} \).

Ex: \( G_2 \)
Approximate Minimum Spanning Tree

Weights \{1, 2, ..., W\}

Let $C_1 = \text{number CC in } G_1$.

Let $C_2 = \text{number CC in } G_2$.

\[\vdots\]

Let $C_j = \text{number CC in } G_j$.

Ex: $G_2$
Claim: \( \text{MST}(G) \) contains \( C_j - 1 \) edges of weight > \( j \).
Claim:
MST(G) contains $C_j - 1$ edges of weight $> j$.

Why?
There are $C_j$ connected components in $G_j$. There must be $C_j - 1$ edges connecting them, and each must have weight $> j$. 

Ex: $G_2$
Approximate Minimum Spanning Tree

Weights \{1, 2, \ldots, W\}

Lemma:

\[
\text{MST}(G) = n - W + \sum_{j=1}^{W-1} C_j
\]

Ex: \(G_2\)
Approximate Minimum Spanning Tree

Weights \{1, 2, \ldots, W\}

Edges of weight 1:

- \(n - 1\) edges total in MST
- \(C_1 - 1\) edges of weight > 1

\[ (n - 1) - (C_1 - 1) \] edges of weight 1.

\[ (n - C_1) \] edges of weight 1.

Ex: \(G_2\)
Approximate Minimum Spanning Tree

Weights \{1, 2, \ldots, W\}

Edges of weight \(j+1\):

\[ C_j - 1 \] edges of weight > \( j \)
\[ C_{j+1} - 1 \] edges of weight > \( j+1 \)

\[ (C_j - 1) - (C_{j+1} - 1) \] edges of weight \( j+1 \).

\[ (C_j - C_{j+1}) \] edges of weight \( j+1 \).

Ex: \( G_2 \)

Note: \( C_j \geq C_{j+1} \)
Approximate Minimum Spanning Tree

Weights \{1, 2, ..., W\}

Sum the weights:

\[
\text{MST}(G) = (n - C_1) + \sum_{j=1}^{W-1} (j + 1)(C_j - C_{j+1})
\]

Note: sum is from \( j = 1 \) to \( W - 1 \).
Approximate Minimum Spanning Tree

Weights \{1, 2, \ldots, W\}

Sum the weights:

\[
\text{MST}(G) = (n - C_1) + \sum_{j=1}^{W-1} (j + 1)(C_j - C_{j+1})
\]

\[
= (n - C_1) + (2C_1 - 2C_2) + (3C_2 - 3C_3) + (4C_3 - 4C_4) + \ldots + (WC_{W_1} - WC_W)
\]
Approximate Minimum Spanning Tree

Weights \{1, 2, ..., W\}

Sum the weights:

\[
\text{MST}(G) = (n - C_1) + \sum_{j=1}^{W-1} (j + 1)(C_j - C_{j+1})
\]

\[
= (n - C_1) + (2C_1 - 2C_2) + (3C_2 - 3C_3) + (4C_3 - 4C_4) + \ldots + (WC_{W-1} - WC_W)
\]

\[
= n + C_1 + C_2 + \ldots + C_{W-1} - WC_W
\]
Approximate Minimum Spanning Tree

Weights \{1, 2, \ldots, W\}

Sum the weights:

$$\text{MST}(G) = n + C_1 + C_2 + \ldots + C_{W-1} - WC_W$$

$$= n + C_1 + C_2 + \ldots + C_{W-1} - W$$

$$= n - W + \sum_{j=1}^{W-1} C_j$$
Approximate Minimum Spanning Tree

Weights \{1, 2, \ldots, W\}

Lemma:

\[ \text{MST}(G) = n - W + \sum_{j=1}^{W-1} C_j \]

Ex: \( G_2 \)
Approximate Minimum Spanning Tree

Algorithm ApproxMST

\[ \text{sum} = n - W \]

for \( j = 1 \) to \( W - 1 \):

\[ X_j = \text{AproxCC}(G_j, d, \varepsilon', \delta) \]

\[ \text{sum} = \text{sum} + X_j \]

return \( \text{sum} \)

Ex: \( G_2 \)
Approximate Minimum Spanning Tree

Error Calculation

\[
\text{sum} = n - W \\
\text{for } j = 1 \text{ to } W - 1:\ \\
X_j = \text{AproxCC}(G_j, d, \varepsilon', \delta) \\
\text{sum} = \text{sum} + X_j \\
\text{return sum}
\]

Set: \( \varepsilon' = \varepsilon/W \)

Sum of errors: \( \leq W(\varepsilon n/W) \leq \varepsilon n \)
Approximate Minimum Spanning Tree

Error Calculation

\[
\text{sum} = n - W \\
\text{for } j = 1 \text{ to } W - 1:\ \\
\quad X_j = \text{AproxC}\text{C}(G_j, d, \varepsilon', \delta) \\
\quad \text{sum} = \text{sum} + X_j \\
\text{return sum}
\]

Guarantee for each AproxC\text{C}:

\[
\Pr \left\{ |X_j - C_j| > \epsilon n / W \right\} < 1/3
\]
Approximate Minimum Spanning Tree

Error Calculation

\[
\text{sum} = n - W \\
\text{for } j = 1 \text{ to } W - 1:\ \\
\quad X_j = \text{AproxCC}(G_j, d, \varepsilon', \delta) \\
\quad \text{sum} = \text{sum} + X_j \\
\text{return sum}
\]

Guarantee for each AproxCC:

\[
\Pr \{ |X_j - C_j| > \epsilon n / W \} < 1/3
\]

Not good enough: \( \Pr\{\text{all correct}\} \approx (2/3)^W \)
Approximate Minimum Spanning Tree

Error Calculation

\[
\text{sum} = n - W \\
\text{for } j = 1 \text{ to } W - 1:\ \\
\quad X_j = \text{AproxCC}(G_j, d, \varepsilon', \delta) \\
\quad \text{sum} = \text{sum} + X_j \\
\text{return } \text{sum}
\]

Set \( \varepsilon' = \varepsilon/W, \delta = 1/(3W) \)

Error probability: \( \Pr \{\text{any fails}\} \leq \sum_{j=1}^{W-1} \frac{1}{3W} \leq \frac{W - 1}{3W} < 1/3 \)
Approximate Minimum Spanning Tree

Error Calculation

\[
\text{sum} = n - W \\
\text{for} \ j = 1 \text{ to } W - 1:\ \\
\text{X}_j = \text{AproxCC}(G_j, d, \epsilon', \delta) \\
\text{sum} = \text{sum} + \text{X}_j \\
\text{return sum}
\]

Set \( \epsilon' = \epsilon/W, \ \delta = 1/(3W) \)

Guarantee for each AproxCC:

\[
\Pr \{|X_j - C_j| > \epsilon n/W \} < \frac{1}{3W}
\]
Approximate Minimum Spanning Tree

Error Calculation

\[ \text{sum} = n - W \]

for \( j = 1 \) to \( W - 1 \):
\[ X_j = \text{AproxCC}(G_j, d, \varepsilon', \delta) \]
\[ \text{sum} = \text{sum} + X_j \]

return \( \text{sum} \)

Set: \( \varepsilon' = \varepsilon/W, \ \delta = 1/(3W) \)

Sum of errors: \( \leq W(\varepsilon n/W) \leq \varepsilon n \)

\[ \Rightarrow \text{MST}(G) - \varepsilon n \leq \text{sum} \leq \text{MST}(G) + \varepsilon n \]
Approximate Minimum Spanning Tree

Error Calculation

\[ \text{MST}(G) \geq n - 1 \geq n/2 \]
Approximate Minimum Spanning Tree

Error Calculation

\[ \text{MST}(G) \geq n - 1 \geq n/2 \]

\[ \text{MST}(G) - \varepsilon n \leq \text{sum} \leq \text{MST}(G) + \varepsilon n \]
Approximate Minimum Spanning Tree

Error Calculation

\[ \text{MST}(G) \geq n - 1 \geq n/2 \]

\[ \text{MST}(G) - \epsilon n \leq \text{sum} \leq \text{MST}(G) + \epsilon n \]

\[ \text{MST}(G) + \epsilon n \leq \text{MST}(G) + \epsilon (2\text{MST}(G)) \leq \text{MST}(G)(1 + 2\epsilon) \]
Approximate Minimum Spanning Tree

Error Calculation

\[ \text{MST}(G) \geq n - 1 \geq n/2 \]

\[ \text{MST}(G) - \epsilon n \leq \text{sum} \leq \text{MST}(G) + \epsilon n \]

\[ \text{MST}(G) + \epsilon n \leq \text{MST}(G) + \epsilon (2\text{MST}(G)) \leq \text{MST}(G)(1 + 2\epsilon) \]

\[ \text{MST}(G) - \epsilon n \geq \text{MST}(G) - \epsilon (2\text{MST}(G)) \geq \text{MST}(G)(1 - 2\epsilon) \]
**Approximate Minimum Spanning Tree**

**Error Calculation**

\[
\text{MST}(G) \geq n - 1 \geq n/2
\]

\[
\text{MST}(G) - \epsilon n \leq \text{sum} \leq \text{MST}(G) + \epsilon n
\]

\[
\text{MST}(G) + \epsilon n \leq \text{MST}(G) + \epsilon (2\text{MST}(G)) \\
\leq \text{MST}(G)(1 + 2\epsilon)
\]

\[
\text{MST}(G) - \epsilon n \geq \text{MST}(G) - \epsilon (2\text{MST}(G)) \\
\geq \text{MST}(G)(1 - 2\epsilon)
\]

\[
\text{MST}(G)(1 - 2\epsilon) \leq \text{MST}(G) \leq \text{MST}(G)(1 + 2\epsilon)
\]
Approximate Minimum Spanning Tree

Running time

```
sum = n - W
for j = 1 to W - 1:
    X_j = AproxCC(G_j, d, \varepsilon', \delta)
    sum = sum + X_j
return sum
```

Set \( \varepsilon' = \varepsilon / W, \delta = 1 / (3W) \)

Running time: \( O \left( W \cdot \frac{d \ln \left( \frac{1}{1/3W} \right)}{(\varepsilon/W)^3} \right) \)
Approximate Minimum Spanning Tree

Running Time

\[
\text{sum} = n - W \\
\text{for } j = 1 \text{ to } W - 1: \quad X_j = \text{AproxCC}(G_j, d, \epsilon', \delta) \\\n\text{sum} = \text{sum} + X_j \\\n\text{return sum}
\]

Set \( \epsilon' = \epsilon/W, \delta = 1/(3W) \)

Running time: \( O \left( W \cdot \frac{d \ln (1/(1/3W)))}{(\epsilon/W)^3} \right) = O \left( \frac{dW^4 \log W}{\epsilon^3} \right) \)
We have shown:

With probability $> \frac{2}{3}$, output is equal to: $\text{MST}(G)(1 \pm \varepsilon n)$

Running time:

$O\left(\frac{dW^4 \log W}{\varepsilon^3}\right)$
Impossible to do better than:

$$\Omega \left( \frac{dW}{\varepsilon^2} \right)$$

Best known:

$$O \left( \frac{dW}{\varepsilon^2 \log \frac{dW}{\varepsilon}} \right)$$

Note: See: Chazelle, Rubinfeld, Trevisan
Summary

Last Week:

Toy example 1: array all 0’s?
• Gap-style question:
  All 0’s or far from all 0’s?

Toy example 2: Faction of 1’s?
• Additive $\pm \varepsilon$ approximation
• Hoeffding Bound

Is the graph connected?
• Gap-style question.
• $O(1)$ time algorithm.
• Correct with probability $2/3$.

Today:

Number of connected components in a graph.
• Approximation algorithm.

Weight of MST
• Approximation algorithm.

Is the graph connected?
Today’s Problem: Maximum Matching

Matching:
Output set of edges $M$ such that no two edges in $M$ are adjacent.

Size of Maximum Matching:
Output the largest value $v$ where there is a matching $M$ of size $v$.

Example:
Size of matching: 5
Maximal Matching:

Output set of edges $M$ such that no two edges in $M$ are adjacent, and no more edges can be added to $M$.

Size of Maximal Matching:

Output the largest value $v$ where there is a maximal matching $M$ of size $v$.

Example:
Size of matching: 5
Today’s Problem: Maximal Matching

**Size of Maximal Matching:**

Output the largest value \( v \) where there is a maximal matching \( M \) of size \( v \).

**Note:**
The maximum matching is at most twice as big as the maximal matching.

\( \rightarrow \)

Maximal is a 2-approximation of maximum.

Example:
Size of matching: 5
Algorithm for maximal matching:

1) Assign each edge a random number. (Equivalent: choose a random permutation of the edges.)
Algorithm for maximal matching:

1) Assign each edge a random number. (Equivalent: choose a random permutation of the edges.)

2) Greedily, in order, try to add each edge to the matching.
Algorithm for maximal matching:

1) Assign each edge a random number. (Equivalent: choose a random permutation of the edges.)

2) Greedily, in order, try to add each edge to the matching.
Algorithm for maximal matching:

1) Assign each edge a random number. *(Equivalent: choose a random permutation of the edges.)*

2) Greedily, in order, try to add each edge to the matching.

⇒ Each random permutation defines a unique maximal matching.
Today’s Problem: Maximal Matching

To solve via sampling:

1) Choose a random permutation for the edges (e.g., a hash function).

2) Choose $s$ edges at random.

3) Decide if they are in the matching for the chosen permutation.
Today’s Problem: Maximal Matching

To decide if an edge is in the matching:

query(e):
    for all neighbors e' of e:
        if query(e') = true
            return false
    return true
Today’s Problem: Maximal Matching

To decide if an edge is in the matching:

query(e):
  for all neighbors e’ of e:
    if query(e’) = true
      return false
  return true

Oops… That doesn’t exactly work!
Today’s Problem: Maximal Matching

To decide if an edge is in the matching:

query(e):
    for all neighbors e’ of e:
        if hash(e’) < hash(e)
            if query(e’) = true
                return false
    return true

hash(e) returns the number chosen for edge e. Only query smaller edges. Larger edges do not matter.
Today’s Problem: Maximal Matching

To decide if an edge is in the matching:

query(e):
    for all neighbors e’ of e:
        if hash(e’) < hash(e)
            if query(e’) = true
                return false
    return true

hash(e) returns the number chosen for edge e. Only query smaller edges. Larger edges do not matter.
Today’s Problem: Maximal Matching

To decide if an edge is in the matching:

\[ \text{query}(e): \]

for all neighbors \( e' \) of \( e \):

\[
\begin{align*}
\text{if } \text{hash}(e') &< \text{hash}(e) \\
\text{if } \text{query}(e') = \text{true} &\text{ return false} \\
\text{return true}
\end{align*}
\]

\text{hash}(e) \text{ returns the number chosen for edge } e.
Only query \textit{smaller} edges. \textit{Larger} edges do not matter.
Today’s Problem: Maximal Matching

To decide if an edge is in the matching:

query(e):
   for all neighbors e’ of e:
      if hash(e’) < hash(e)
       if query(e’) = true
          return false
      return true

return true

hash(e) returns the number chosen for edge e. Only query smaller edges. Larger edges do not matter.
Today’s Problem: Maximal Matching

To decide if an edge is in the matching:

query(e):
  for all neighbors e’ of e:
    if hash(e’) < hash(e)
      if query(e’) = true
        return false
    return true

hash(e) returns the number chosen for edge e.
Only query smaller edges. Larger edges do not matter.
Today’s Problem: Maximal Matching

To decide if an edge is in the matching:

query(e):
for all neighbors e’ of e:
if hash(e’) < hash(e)
  if query(e’) = true
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hash(e) returns the number chosen for edge e. Only query smaller edges. Larger edges do not matter.
Today’s Problem: Maximal Matching

To decide if an edge is in the matching:

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Key question: How expensive is a query?

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Some simple analysis:

If graph has maximum degree $d$, then there are at most $2d^k$ paths of length $k$ starting from the query edge.
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Permutation: $[6, 1, 11, 10, 3]$
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There are \(k!\) possible permutations.

\[
\text{Pr[ path is all decreasing ]} = \frac{1}{k!}
\]

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Today’s Problem: Maximal Matching

Conclusion:

The expected number of paths traversed of length $k$ is at most: $\frac{d^k}{k!}$

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The expected total cost of a query is:

$$\sum_{k=1}^{\infty} \frac{d^k}{k!} = O(e^d)$$

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Key question:
How expensive is a query?

\[ \mathbb{E}[\text{cost}] = O(e^d) \]

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Today’s Problem: Maximal Matching

To solve via sampling:

1) Choose a random permutation for the edges (e.g., a hash function).

2) Choose $s$ edges at random.

3) Decide if they are in the matching for the chosen permutation via query operation.
Approximate Maximal Matching

MaxMatch-Sampling

\[
\text{sum} = 0 \\
\text{for } j = 1 \text{ to } s: \\
\quad \text{Choose edge } e \text{ uniformly at random.} \\
\quad \text{if } (\text{query}(e) = \text{true}) \text{ then} \\
\quad \quad \text{sum} = \text{sum} + 1 \\
\text{return } m \cdot (\text{sum}/s)
\]
Approximate Maximal Matching

MaxMatch-Sampling

```
sum = 0
for j = 1 to s:
    Choose edge e uniformly at random.
    if (query(e) = true) then
        sum = sum + 1
return m \cdot (sum/s)
```

Claim: returns size of maximal matching $\pm \epsilon m$
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return m \cdot (\text{sum/s})

Claim: returns size of maximal matching \( \pm \varepsilon m \)

Claim: Runs in time \( O(e^d / \varepsilon^2) \)
Today’s Problem: Maximal Matching

Two improvements:

1) Reduce error from $\pm \varepsilon m$ to $\pm \varepsilon n$. 
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   (Hint: each node is either matched or unmatched, 
   and you can compute the size of the matching 
   from the number of matched nodes.)
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1) Reduce error from $\pm \varepsilon m$ to $\pm \varepsilon n$. (Hint: each node is either matched or unmatched, and you can compute the size of the matching from the number of matched nodes.)

2) Reduce the running time from exponential to $O(d^4/\varepsilon^2)$. 
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1) Reduce error from $\pm \epsilon m$ to $\pm \epsilon n$.  
   (Hint: each node is either matched or unmatched, and you can compute the size of the matching from the number of matched nodes.)

2) Reduce the running time from exponential to $O(d^4/\epsilon^2)$.  
   (Hint: In query, explore neighboring edges in order of smallest weight first. Analysis is not simple!)
Questions to think about:

1) Show that the sampling algorithm works as claims (if the query operation is correct).

2) Reduce error from $\pm \varepsilon m$ to $\pm \varepsilon n$.
   (Hint: each node is either matched or unmatched, and you can compute the size of the matching from the number of matched nodes.)

3) Can you find a multiplicative (instead of additive) approximation? Why not?
   (Hint: Think about a graph where the maximal matching is very small.)
Two more questions:

1) Give an algorithm for deciding if the black pixels are connected or \( \epsilon \)-far from connected in an \( n \) by \( n \) square of pixels.

2) Give an algorithm for deciding if the black pixels are a rectangle or \( \epsilon \)-far from a rectangle in an \( n \) by \( n \) square of pixels.

*Hint: imagine querying a grid of pixels distance \( \epsilon n \) apart.*
Summary

Last Week:

Toy example 1: array all 0’s?
• Gap-style question:
  All 0’s or far from all 0’s?

Toy example 2: Faction of 1’s?
• Additive $\pm \varepsilon$ approximation
• Hoeffding Bound

Is the graph connected?
• Gap-style question.
• $O(1)$ time algorithm.
• Correct with probability $2/3$.

Today:

Number of connected components in a graph.
• Approximation algorithm.

Weight of MST
• Approximation algorithm.

Size of maximal matching
• Approximation algorithm.