

Subject:

Date:

Markov Chains & Random Walks

Overview

- 1) Intro to random walks
- 2) Pass the Candy & analyzing a simple random walk
- 3) 2 SAT (and 3 SAT)
- 4) Markov Chains
- 5) Undirected Graphs

Random Walks

- robot randomly exploring a maze
- drunk man leaves a bar → randomly stumbles in some direction

$P(\text{arrives home safely})?$
 $E(\text{time to get home})?$

Surprising fact Drunk man eventually arrives home
 Drunk bird does not!

2-d world

3-d world

Other examples:

physics → gas diffusion

biology → population dynamics

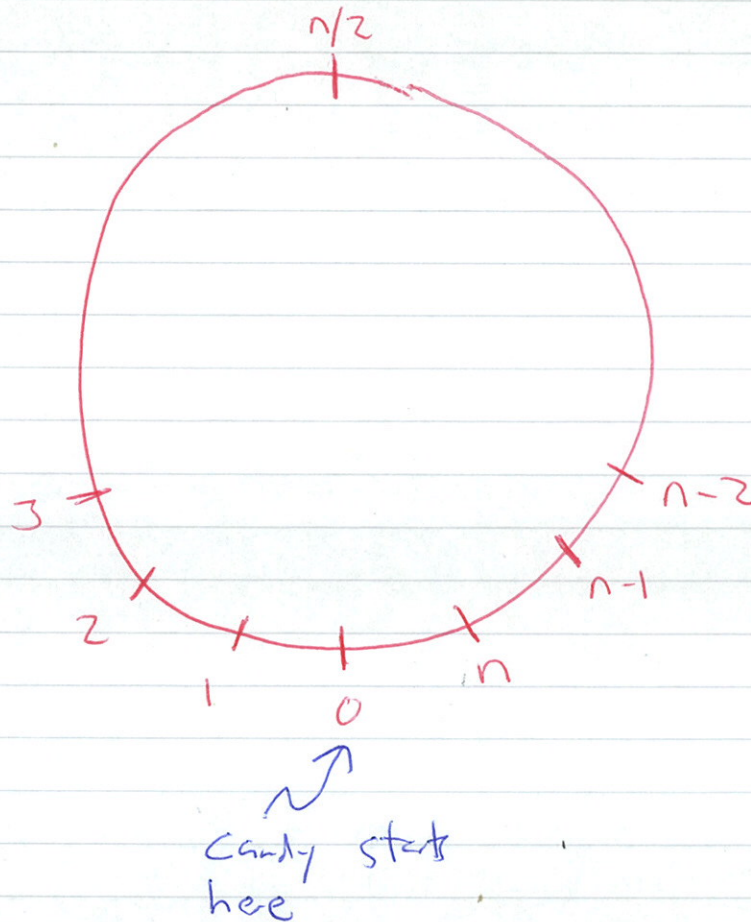
finance → market price fluctuations

CS → Google

pagerank = how frequently a random walk visits a webpage

Pass The Candy

- n students
- 1 professor
- 1 big bowl of candy



Rules:

- take 1 piece of candy
- flip a coin: $P(1/2)$
- if heads, pass left
else pass right

Who is the last to get candy?

Maybe $n/2$?

Maybe 1 or n ?

Variant $\rightarrow$ Line

A S_1 S_2 S_3 $\dots$ S_K B
 $\uparrow$
 candy

What is the probability that A gets candy before B gets candy?

$$\Pr(A \text{ before } B) = P_K \quad \leftarrow K \text{ people}$$

Example: $K=1$

A S_1 B
 $\uparrow$
 candy

$$P_1 = 1/2$$

General case:

$$P_K = \Pr(\text{left}) \cdot \Pr(A \rightarrow B | \text{left}) + \Pr(\text{right}) \cdot \Pr(A \rightarrow B | \text{right})$$

$$= \left(\frac{1}{2}\right) \cdot 1 + \left(\frac{1}{2}\right) \cdot \Pr(A \rightarrow B | \text{right})$$

$\uparrow$
 what is this?

$$\Pr(S_1 \rightarrow B) \cdot P_K$$

A S_1 S_2 S_3 $\dots$ S_K B
 $\uparrow$
 candy here

$$Pr(S_1 \rightarrow B) = P_{K-1} \leftarrow \text{same problem, } K-1 \text{ people}$$

$$P_K = 1/2 + (1/2) P_{K-1} \circ P_K$$

$$\Rightarrow 2P_K - P_{K-1}P_K = 1$$

$$\Rightarrow P_K = \frac{1}{2 - P_{K-1}}$$

Base case: $P_1 = 1/2$

Try a few examples:

$$P_1 = 1/2$$

$$P_2 = 2/3$$

$$P_3 = 1/\frac{6}{3} - \frac{2}{3} = 3/4$$

$$P_4 = 1/\frac{8}{4} - \frac{3}{4} = 4/5$$

Guess: $P_K = \frac{K}{K+1}$

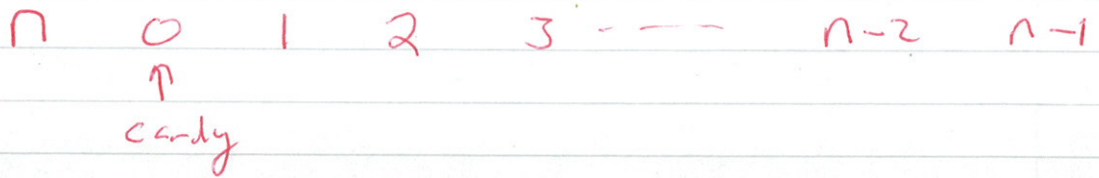
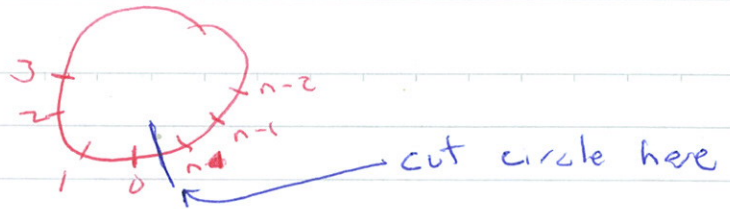
Verify: $P_{K+1} = \frac{1}{2 - P_K} = \frac{1}{2 - \frac{K}{K+1}}$

$$= \frac{1}{\frac{2K+2-K}{K+1}} = \frac{1}{\frac{K+2}{K+1}}$$

$$= \frac{K+1}{K+2}$$



For $K = n-1$: $P_K = \frac{n-1}{n}$

Candy Game

If $n-1$ gets candy before n , then n is last!

$$Pr(n \rightarrow n-1) = P_{n-1} = \frac{n-1}{n}$$

$$Pr(n \text{ is last}) = 1 - \frac{n-1}{n} = \frac{1}{n}$$

← Surprising?
 n is not
 least likely to be
 last!

Symmetry:

$$Pr(1 \text{ is last}) = \frac{1}{n}$$

Player i is last

$$Pr(i \text{ is last}) = Pr(i+1 \rightarrow i-1) \cdot Pr(i \text{ last} | i+1 \rightarrow i-1) + Pr(i-1 \rightarrow i+1) \cdot Pr(i \text{ last} | i-1 \rightarrow i+1)$$



$$Pr(i \text{ last} | i+1 \rightarrow i-1) = 1 - P_{n-1} = \frac{1}{n}$$

$$Pr(i \text{ last} | i-1 \rightarrow i+1) = 1 - P_{n-1} = \frac{1}{n}$$

$$Pr(i \text{ is last}) = Pr(i+1 \rightarrow i-1) \cdot \frac{1}{n} + Pr(i-1 \rightarrow i+1) \cdot \frac{1}{n} = \frac{1}{n} (Pr(i+1) + Pr(i-1)) = \frac{1}{n}$$

Conclusion

Each player is equally likely to be last.

$$P(j \text{ is last}) = 1/n$$

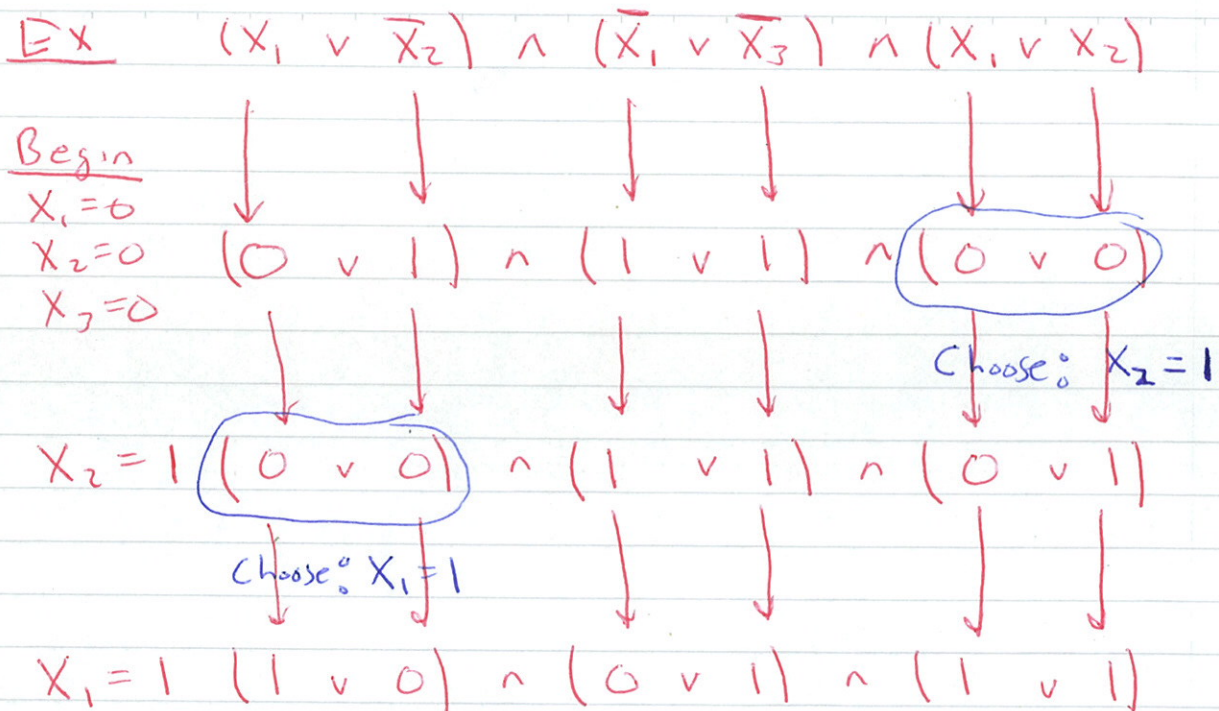
How long does it take?

2 SAT: EX $(x_1 \vee \bar{x}_2) \wedge (\bar{x}_1 \vee \bar{x}_3) \wedge (x_1 \vee x_2) \dots$

→ deterministic, linear time algorithm
(somewhat complicated)

Algorithm

- 1) Choose arbitrary values for x_j
constant
- 2) Repeat $2Kn^2$ times:
variables
 - a) Choose arbitrary clause that is not satisfied
 - b) Choose one of the literals uniformly and switch it.
- 3) If satisfied, return assignment.
Else return UNSAT



Claim 1 If Φ unsat, then returns UNSAT

Harder If Φ sat, then returns assignment

Let A = satisfying assignment (any one)

Let B_j = algorithm assignment on step j

Let $X_j = \#$ variables equal in A and B_j

Ex: $A = (1, 1, 0)$

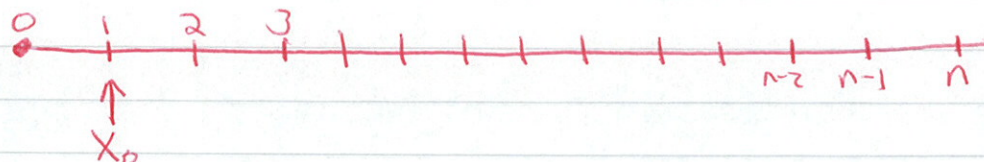
$X_1 = 1$	0, 0, 0
$X_2 = 2$	0, 1, 0
$X_3 = 3$	1, 1, 0

$A = (1, 0, 0)$

$X_1 = 2$	0, 0, 0
$X_2 = 1$	0, 1, 0
$X_3 = 2$	1, 1, 0

Fact $X_0 \geq 0$

Fact if $X_m = n$, then sat found $\Rightarrow B_m = A$



Assume $X_t = j$.

What is X_{t+1} ?

If both disagree with A, then $\Pr(j+1) = 1$

$$\Pr(X_{t+1} = j+1 \mid X_t = j) \geq \frac{1}{2}$$

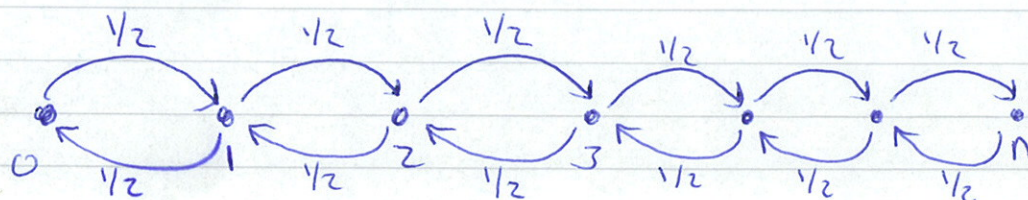
$$\Pr(X_{t+1} = j-1 \mid X_t = j) \leq \frac{1}{2}$$

Example

$(X_7 \vee \bar{X}_9) \leftarrow$ bad clause

Either X_7 or $\bar{X}_9$ is different from A, since A is sat.

Fix "right" variable w. prob. $\geq \frac{1}{2}$



If random walk starts at 0, how long until it reaches n?

Note: dominates real algorithm

Answer $O(n^2)$ steps, in expectation

$\Rightarrow$ Alg. correct w.p. $1 - 1/2^k$

See textbook for direct calculation.

(Similar to "pass the candy": define recurrence and solve.)

Today's general technique

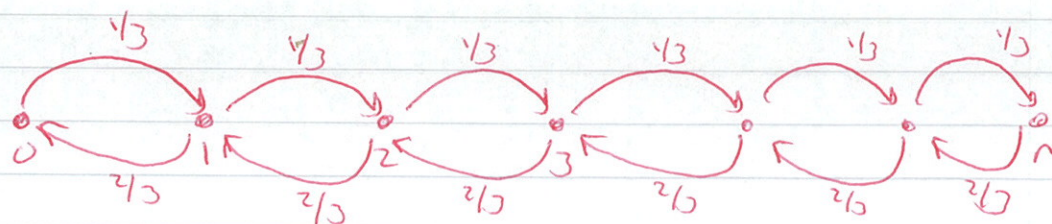
Aside Why not 3SAT?

Repeat

- Choose UNSAT clause
- Choose literal uniformly and fix it.

Problem $\Pr(X_{t+1} = j+1 \mid X_t = j) \geq 1/3$

$\Pr(X_{t+1} = j-1 \mid X_t = j) \leq 2/3$



How long?

Show: $\Theta(2^n)$ steps to reach n

Idea

Repeat:

if lucky,
lots of sat
clauses

- Choose assignment at random
- Random walk for $3n$ steps

explore nearby
assignments

$$\Rightarrow E[\# \text{ steps}] = O(n^{3/2} (\frac{4}{3})^n)$$

Markov Chains

(discrete)

Random variables: $X_1, X_2, X_3, \dots$ Ex $X_t = \# \text{ matching variables in } A \text{ w/ } B_t$
at time t Markov chain

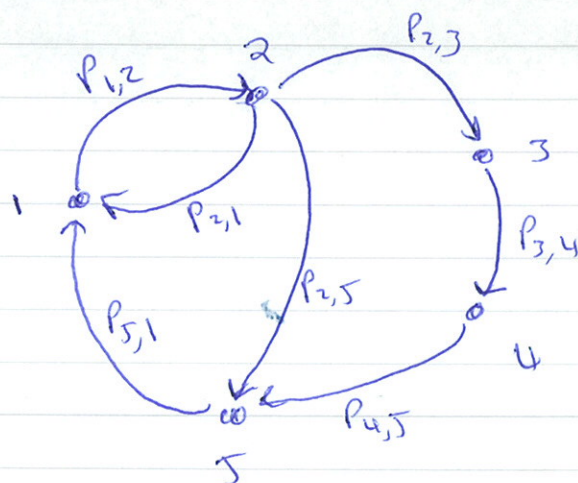
$$Pr(X_{t+1} | X_1, X_2, \dots, X_t) = Pr(X_{t+1} | X_t)$$

only last state
matters!

"memoryless"

(not "independent")

Define $P_{i,j} = Pr(X_t = j | X_{t-1} = i)$

= move from i to j Graph view

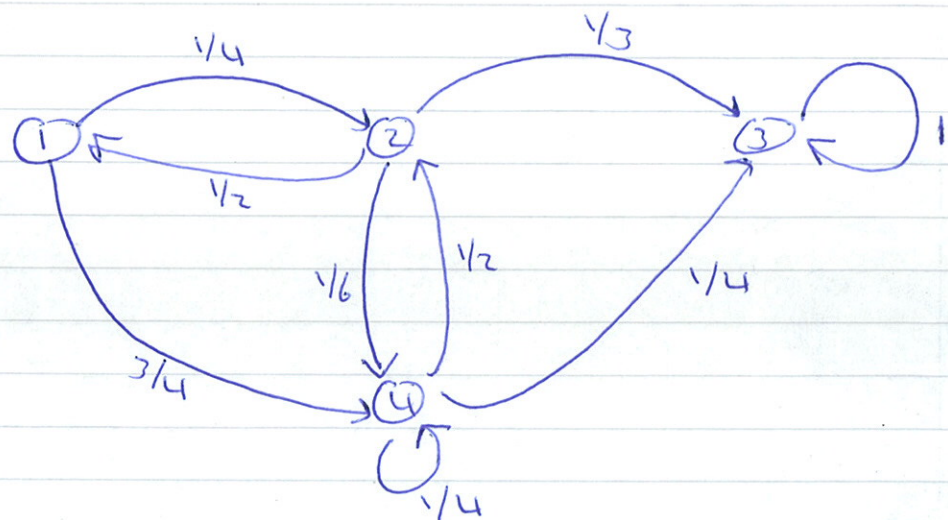
- directed graph

- edges = probability
(i, j)- $X_t = \text{location in}$
graph at
time t Markov Chain = random walk
on graph

Matrix view

$$P = \begin{matrix} & \begin{matrix} 1 & 2 & 3 & 4 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix} & \begin{bmatrix} 0 & 1/4 & 0 & 3/4 \\ 1/2 & 0 & 1/3 & 1/6 \\ 0 & 0 & 1 & 0 \\ 0 & 1/2 & 1/4 & 1/4 \end{bmatrix} \end{matrix}$$

$$P_{i,j} = P(i \rightarrow j)$$



Let $V = \text{distribution}$

e.g. $V = [1/2, 1/2, 0, 0]$

$\Rightarrow VP = \text{distribution after 1 step}$

$$[1/2, 1/2, 0, 0] \begin{bmatrix} 0 & 1/4 & 0 & 3/4 \\ 1/2 & 0 & 1/3 & 1/6 \\ 0 & 0 & 1 & 0 \\ 0 & 1/2 & 1/4 & 1/4 \end{bmatrix} =$$

$$[1/4, 1/8, 1/6, 1/24]$$

$\Rightarrow VP^K = \text{distribution after } K \text{ steps}$

Stationary Distribution

$$\boxed{\pi P = \pi}$$

← π is stationary distribution

"equilibrium"

Ex $P_i(X_t) = \pi$

$$P_i(X_{t+1}) = \pi P = \pi$$

Three

~~Two~~

questions:

1) What is the stationary distribution?

2) How long to converge to the stationary distribution?

0) Does a stationary distribution exist?

"Stationary" = long run behavior

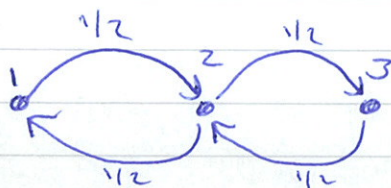
How often is the random walk in state v ?

Terminology

"Irreducible" $\Leftrightarrow$ graph is strongly connected
 "any 2 states are reachable from each other"

"Periodic" $\Leftrightarrow$ for some Δ

$$P(X_{t+s}=j | X_t=j) = 0 \text{ unless } s \text{ is divisible by } \Delta$$

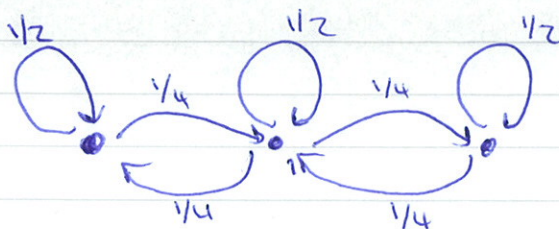
Ex

$$P(X_2=1 | X_1=1) = 0$$

no way to return to 1 in an odd number of steps!

Bipartite graphs are periodic

Solution Self-loops



$\Rightarrow$ aperiodic

Fundamental Theorem of Markov Chains

If X is a finite, irreducible, aperiodic Markov Chain, then

1) There exists a unique stationary distribution π

$$2) \pi_i = \lim_{t \rightarrow \infty} P_{ji}^t, \text{ for all } j$$

$\uparrow$ $P_i(\text{go from } j \rightarrow i \text{ in } t \text{ steps})$

$$3) \pi_i = \frac{1}{h_{ii}}$$

$\uparrow$ $E[\# \text{ steps start at } i, \text{ return to } i \text{ for 1st time}]$

Subject: _____

No: _____

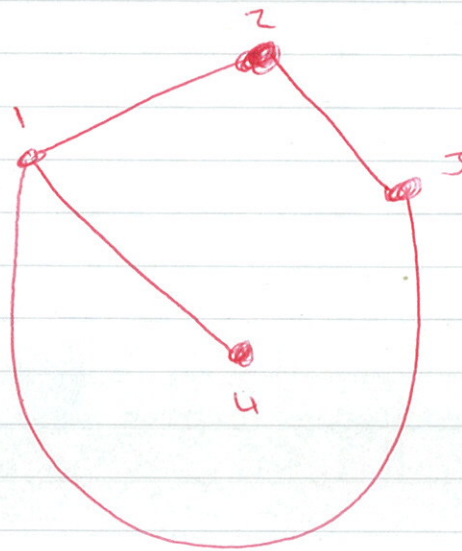
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Random Walk on Undirected Graph

Let $G = (V, E)$ be a finite
undirected
connected
not-bipartite

← add self-loops

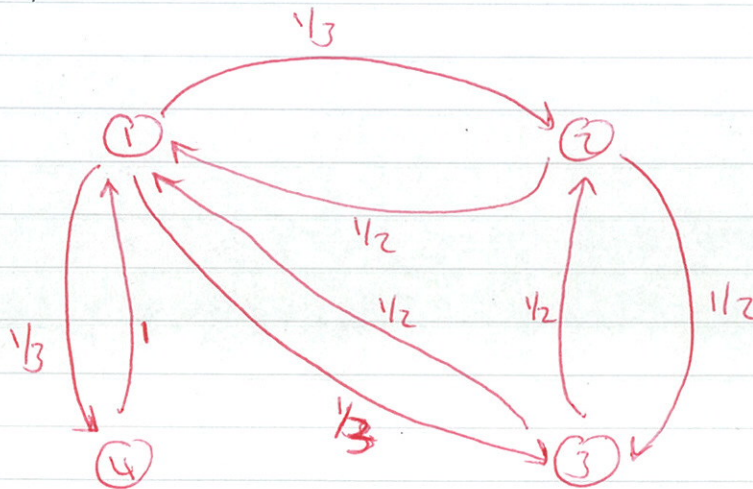
Ex



Questions:

How long to return home?

How long to "cover" graph?



$$\begin{bmatrix} 0 & 1/3 & 1/3 & 1/3 \\ 1/2 & 0 & 1/2 & 0 \\ 1/2 & 1/2 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

$$P_{ij} = \frac{1}{d(i)}$$

Random walk on G_0 $d(v) = \text{degree } v$

$$\Pr(X_{t+1} = w \mid X_t = v) = \begin{cases} 0 & \text{if } (v, w) \notin E \\ 1/d(v) & \text{if } (v, w) \in E \end{cases}$$

- finite (by assumption)
- aperiodic (not bipartite)
- irreducible (connected)

$$\begin{aligned} \implies \pi \text{ exists} \\ X \text{ converges to } \pi \\ \pi_v = 1/h_{v,v} \end{aligned}$$

Claim $\pi_v = \frac{d(v)}{2|E|}$

Note: $\sum_{j=1}^n \pi_j = \sum_{j=1}^n \frac{d(j)}{2|E|} = \frac{2|E|}{2|E|} = 1$

Pf π exists and is unique

Verify ~~$\pi P = \pi$~~ $\pi P = \pi$

$\implies \pi$ is s.d.

$$\begin{bmatrix} \pi \end{bmatrix} \begin{bmatrix} P \end{bmatrix}$$

$$\pi_v = \sum_{j=1}^n \pi_j P_{jv}$$

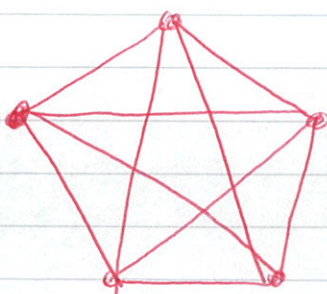
$$= \sum_{j \in N(v)} \pi_j P_{jv} = \sum_{j \in N(v)} \frac{d(j)}{2|E|} \cdot \frac{1}{d(j)}$$

$$= \sum_{j \in N(v)} \frac{1}{2|E|} = \frac{d(v)}{2|E|} \quad \checkmark$$

Corollary $E[\# \text{ steps to return to } v] = \frac{2E}{d(v)}$

$$\frac{d(v)}{2E} = \pi_v = \frac{1}{h_{v,v}}$$

Ex Random walk on K_n



$$h_{o,o} = \frac{2 \binom{n}{2}}{n-1} = O(n)$$

$E[\text{walk from } v \text{ to } u]$

Lemma If $(u, v) \in E$, then $h_{v,u} < 2|E|$

Proof $h_{u,u} = \sum_{w \in N(u)} \Pr(u \rightarrow w) \cdot [E[w \rightarrow u | X=w] + 1]$

law conditional expectation

$$= \frac{1}{d(u)} \sum_{w \in N(u)} (1 + h_{w,u})$$

$$= \frac{2|E|}{d(u)} \quad \leftarrow \text{by Corollary, } \pi = \frac{d(u)}{2|E|}$$

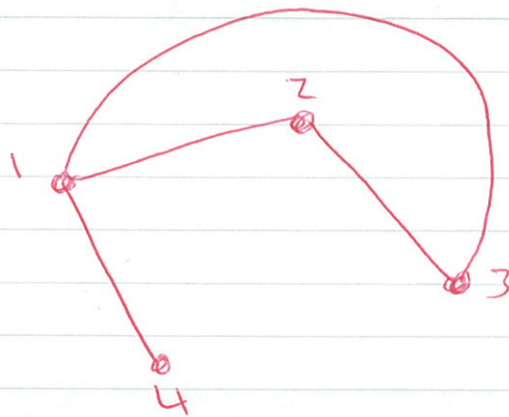
$$\Rightarrow 2|E| = \sum_{w \in N(u)} (1 + h_{w,u})$$

$$\Rightarrow h_{w,u} < 2|E| \text{ for all } w \in N(u)$$

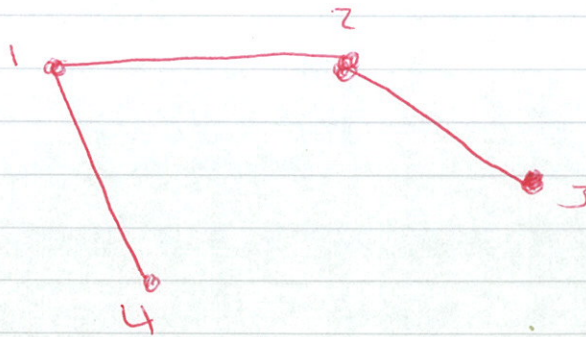
Assume random walk starts at node v .

How long until it visits every node?

↖ "Cover time"



1) Choose any spanning tree $T = (V, E')$



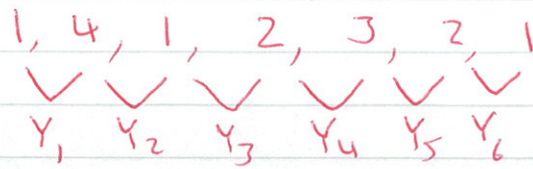
$$\# \text{ edges } = V - 1$$

2) Do DFS on tree

$(1, 4), (4, 1), (1, 2), (2, 3), (3, 2), (2, 1)$

$$\text{Length of DFS} \leq 2|E'| \leq 2V$$

3) Let $Y_j = \# \text{ steps from } (v_j, v_{j+1}) \text{ in DFS}$



$$E[Y_j] = h_{v_j, v_{j+1}} \leq 2E$$

$$E[\text{cover time}] = \sum_{j=1}^{2V} E[Y_j]$$

$$\leq \sum_{j=1}^{2V} 2E$$

$$\leq 4EV$$

length of each step in DFS $\uparrow$ $\nwarrow$ # steps in DFS

Conclusion: $E[\text{cover time}] \leq 4EV$

Corollary For all G , $E[\text{cover}] \leq O(n^3)$

Corollary



$$E[2SAT] \leq 4 \cdot (2n) \cdot n = O(n^2)$$

Modify sls

wp: $1/3$ do nothing

wp: $2/3$ update variable

