HOPI: A Novel High Order Parametric Interpolation in 2D

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Abstract

Our paper proposes a novel smooth and convergent high order interpolation method with rigorous treatment. The new method employs high order derivatives and provides more freedom on control of curve/surface. It can be used to design complex mathematical plane curve and surface.

Contributions

Parametric representation

We come out a new smooth and convergent interpolation method called **HOPI** which exploits the derivative information based on **Taylor Theorem**. Our solution separates the derivative information (eg, normal) from neighborhood providing more freedom for user control and capability to represent very complicate curve/surface. The derivative information could be estimated from neighboring cloud or specified by user. So it can be used in both curve/surface design and curve/surface representation.

Formal Definition of the High Order Interpolation Problem

Definition (HOPI *problem)* Given a set $\mathbf{P}$ of n plane points with derivatives up to order m,

$$\mathbf{P} = \{p_i | 1 \le i \le n, p_i = (x_i, y_i^0, y_i^1, y_i^2 \dots y_i^m)\},$$

where $a = x_1 < x_2 < \cdots < x_n = b$, find a function $F\colon [a,b] \to \mathbf{R}$, such that

$$\forall 1 \le i \le n, \forall 0 \le k \le m, F^{(k)}(x_i) = y_i^k.$$

We denote this problem with $\mathbf{HOPI(n,m)}$, **HOPI** stands for "High Order Parametric Interpolation", n is the number of points and m is the highest order.

We construct the function $F\colon [a,b] \to \mathbf{R}$

$$F(x) = \sum_{i=1}^{n} T_i(x) \cdot I_i(x). \qquad (1)$$

Remark $\mathbf{HOPI(n,0)}$ is the normal historic position interpolation problem. $\mathbf{HOPI(1,m)}$ is resolved by Taylor Theorem.

Remark Here, T_i captures all local information associated with point p_i. And I_i works like a weighting function which specifies the range and extent of influence of T_i on other points.

The underlying principles are, given $\mathbf{P}$ is all information we know about f:

- Each point carries full information we can know of f at that point;
- Each point carries no global information but only local information of f.

Consequently, in our approach, local modifications, like changing, removing or adding points have only local effects.

Interpolation, smoothness and convergence conditions

Interpolation Conditions

In order to satisfy condition (1), we assume functions T_i and functions I_i satisfy the following conditions :

$$\forall 1 \le i \le n, \forall 0 \le k \le m, T_i^{(k)}(x_i) = y_i^k, \qquad (2)$$

$$\forall 1 \le i,j \le n, I_i(x_j) = \delta_{ij}, \qquad (3)$$

$$\forall 1 \le i,j \le n, \forall 1 \le k \le m, I_i^{(k)}(x_j) = 0. \qquad (4)$$

Smoothness Conditions

Theorem 0.1 (Smoothness) *If* $\forall 1 \le i \le n, T_i \in C^{(m)}[a,b]$, *and* $I_i \in C^{(m)}[a,b]$, *then* $F \in C^{(m)}[a,b]$.

Convergence Conditions

We find our previous assumptions (2), (3), (4) are not enough to guarantee the convergence of F. We need more conditions:

$$\forall 1 \le i \le n, \forall x \in [a,b], f(x) = T_i(x) + o\left((x - x_i)^m\right), \qquad (5)$$

$$\forall 1 \le i < n, \forall x \in [a,b], I_i(x) + I_{i+1}(x) = 1, \qquad (6)$$

$$\forall 1 \le i \le n, \forall x \in [a,b], 0 \le I_i(x) \le 1. \qquad (7)$$

Exhausting the principle of localization, we restrict the influence of T_i to its neighborhood.

$$\forall 1 \le i \le n, \forall x \in [a,b] \backslash (x_{i-1}, x_{i+1}),\ I_i(x) = 0, \qquad (8)$$

where $x_0 = x_1, x_{n+1} = x_n$.

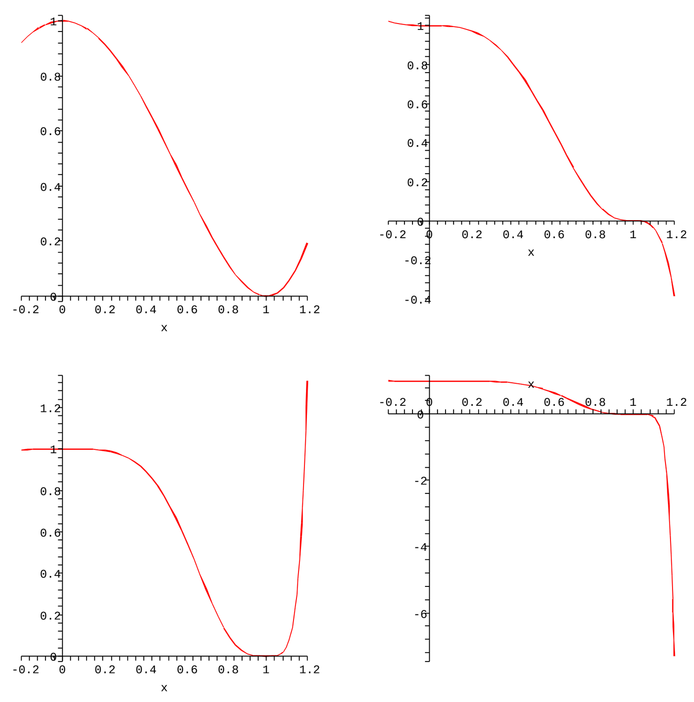
Choice of T and I

Taylor expansion is a good choice of T

Definition *(Function class $\mathcal{A}$)* Define $\mathcal{A}_m$ a set of functions $g\colon [0,1] \to \mathbf{R} \in C^{(m)}[0,1]$, such that

$$g(0) = 1, g(1) = 0;$$
$$\forall 1 \le k \le m, g^{(k)}(0) = g^{(k)}(1) = 0;$$
$$\forall x \in [0,1], g(x) \ge 0.$$

Lemma 0.2 $g(x) = (1 - x^m)^m \in \mathcal{A}_{m-1}$



Graphs of $(1 - x^m)^m$ with m = 2,3,4,5

Definition *(Function class $\mathcal{B}$)* Define $\mathcal{B}$ a set of functions $g\colon [0,1] \to \mathbf{R}$, such that

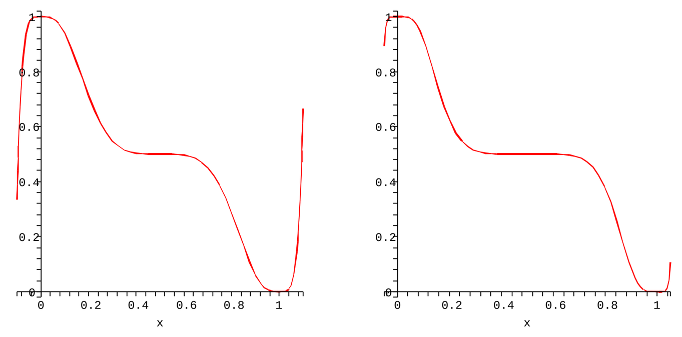
$$\forall x \in [0,1], g(x) + g(1-x) = 1.$$

Definition *(Basis I function of order m)* A basis I function of order m is any function $g\colon [0,1] \to \mathbf{R} \in \mathcal{A}_m \bigcap \mathcal{B}$.

Theorem 0.3 (Candidate 1 of basis I function) *If* $g\colon [0,1] \to \mathbf{R} \in \mathcal{A}_m$, *then* $G\colon [0,1] \to \mathbf{R} \in \mathcal{A}_m \bigcap \mathcal{B}$, *i.e,* G *is a basis* I *function of order* m, *where* G *is defined as follow*:

$$G(x) = \begin{cases} 1 - \frac{1}{2} g(-2x+1) & \text{if } x \in [0, \frac{1}{2}]; \\ \frac{1}{2} g(2x-1) & \text{if } x \in [\frac{1}{2}, 1]. \end{cases}$$

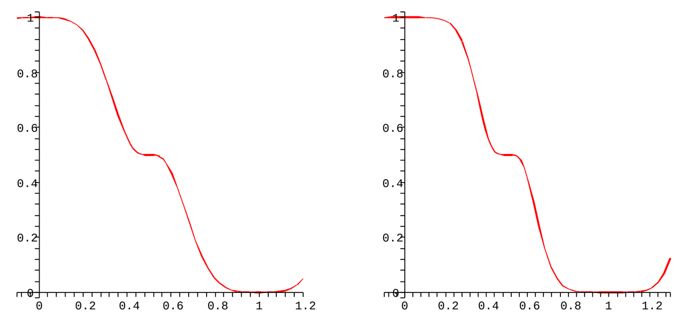
Graphs of G with $g = (1 - x^m)^m$ and m = 4, 6



Theorem 0.4 (Candidate 2 of basis I function) *If* $g\colon [0,1] \to \mathbf{R} \in \mathcal{A}_m$, *then* $H\colon [0,1] \to \mathbf{R} \in \mathcal{A}_m \bigcap \mathcal{B}$, *i.e,* H *is a basis* I *function of order* m, *where* H *is defined as follow*:

$$H(x) = \begin{cases} \frac{1}{2} + \frac{1}{2} g(2x) & \text{if } x \in [0, \frac{1}{2}]; \\ \frac{1}{2} - \frac{1}{2} g(2-2x) & \text{if } x \in [\frac{1}{2}, 1]. \end{cases}$$

Theorem(0.4) is almost identical to **Theorem**(0.3) except the way of transformation.



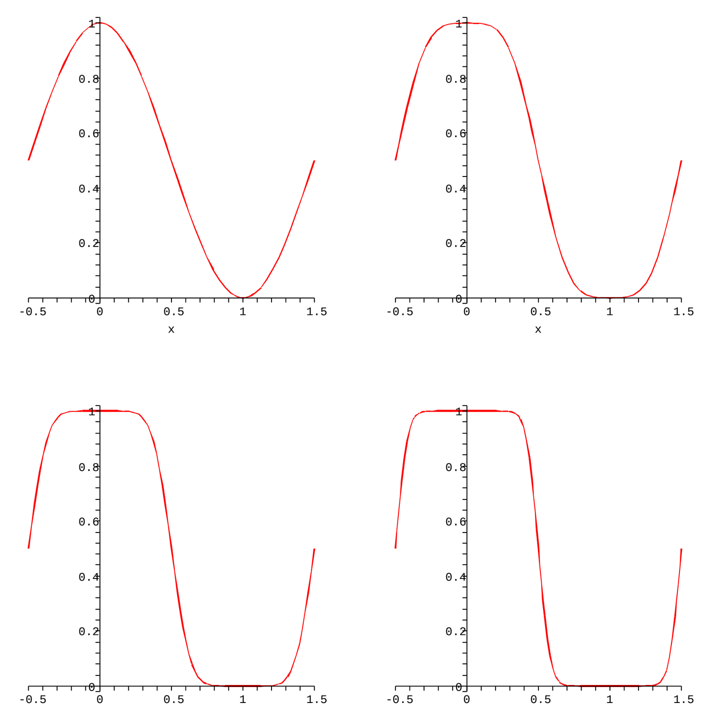
Graphs of H with $g = (1 - x^m)^m$ and $m = 4, 6$

Theorem 0.5 (Iteration) *If* $g \in \mathcal{A}_{m_1} \bigcap \mathcal{B}$, *and* $h \in \mathcal{A}_{m_2} \bigcap \mathcal{B}$ *then* $g(h(1-x)) \in \mathcal{A}_{m_1 m_2 + m_1 + m_2} \bigcap \mathcal{B}$.

Theorem 0.6 (Candidate 3 of basis I function) *Function* $\varphi := \frac{1}{2} + \frac{1}{2}\cos(\pi x) \in \mathcal{A}_1 \bigcap \mathcal{B}$. *And* $\varphi_m \in \mathcal{A}_{2^m - 1} \bigcap \mathcal{B}$. , *where* φ_n *is defined as follow*

$$\varphi_1(x) := \varphi(x),\ \ \varphi_{n+1}(x) := \varphi \circ \varphi_n(1-x).$$

Graphs of φ_n with n = 1, 2, 3,4



Construction of I_i function

If g is a basis I function of order m, then we define I_i of $\mathbf{H(n,m)}$ as follow

$$I_1(x) = \begin{cases} g(\frac{x - x_1}{x_2 - x_1}) & \text{if } x \in [x_1, x_2]; \\ 0 & \text{otherwise.} \end{cases}$$

$$I_n(x) = \begin{cases} g(\frac{x_n - x}{x_n - x_{n-1}}) & \text{if } x \in [x_{n-1}, x_n]; \\ 0 & \text{otherwise.} \end{cases}$$

$$\forall 1 < i < n,\ \ I_i(x) = \begin{cases} g(\frac{x - x_i}{x_{i+1} - x_i}) & \text{if } x \in [x_i, x_{i+1}]; \\ g(\frac{x_i - x}{x_i - x_{i-1}}) & \text{if } x \in [x_{i-1}, x_i]; \\ 0 & \text{otherwise.} \end{cases}$$