HOPI: A Novel High Order Parametric Interpolation in 2D

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Abstract
Our paper proposes a novel smooth and converged high order interpolation method with rigorous treatment. The new method employs high order derivatives and provides more freedom on control of curve/surface. It can be used to design complex mathematical plane curve and surface.

Contributions
Parametric representation We come out a new smooth and converged interpolation method called HOPI which exploits the derivative information based on Taylor Theorem. Our solution separates the derivative information (eg, normal) from neighborhood providing more freedom for user control and capability to represent very complex curve/surface. The derivative information could be estimated from neighboring cloud or specified by user. So it can be used in both curve/surface design and curve/surface representation.

Formal Definition of the High Order Interpolation Problem
Definition (HOPI problem) Given a set P of n plane points with derivatives up to order m, P = {p | 1 ≤ i ≤ n, pi ∈ {x1, x2, ..., xn}}, where x1 < x2 < ... < xn, find a function F : [a, b] → R, such that

\[ F(x) = \sum_{i=1}^{n} p_i \cdot I_i(x), \]

Remark HOPI: n, 0: is the normal historic position interpolation problem. HOPI: 1, m: is resolved by Taylor Theorem.

Remark Here, T captures all information associated with point p. And I works like a weighting function which specifies the range and extent of influence of T on other points. The underlying principles are, given P we know about f:
• Each point carries full information we can know of \( f \) at that point,
• Each point carries no global information but only local information of \( f \).

Consequently, in our approach, local modifications, like changing, removing or adding points have only local effects.

Interpolation, smoothness and convergence conditions
Interpolation Conditions In order to satisfy condition (1), we assume functions \( F \) and functions \( I \) satisfy the following conditions
\[ \forall 1 \leq i \leq n, \sum_{j=1}^{m} \frac{\partial^j F}{\partial x^j}(x_i) = \frac{\partial^j p_i}{\partial x^j}(x_i), \]

\[ \forall 1 \leq i \leq n, \sum_{j=1}^{m} \frac{\partial^j I_i(x)}{\partial x^j}(x_i) = \frac{\partial^j p_i}{\partial x^j}(x_i), \]

\[ \forall 1 \leq i, j \leq n, \forall 1 \leq k \leq m, \frac{\partial^k I_i(x)}{\partial x^k}(x_i) = 0. \]

Smoothness Conditions Theorem 0.1 (Smoothness) \( f \) is a smooth function \( 0 \leq i \leq n, T_i \in C^{m}(a, b), \) and \( I_i \in C^{m}(a, b), \) then \( F \in C^{m}(a, b). \)

Convergence Conditions We find our previous assumptions (2), (3), (4) are not enough to guarantee the convergence of \( F \). We need more conditions:
\[ \forall 1 \leq i \leq n, \sum_{j=1}^{m} \frac{\partial^j p_i}{\partial x^j}(x_i) = 1. \]

Exhausting the principle of localization, we restrict the influence of \( T \) to its neighborhood.
\[ \forall 1 \leq i \leq n, \forall 1 \leq j \leq m, \frac{\partial^j I_i(x)}{\partial x^j}(x_i) = 0, \]

where \( x_i = x_i, y_i, z_i \).

Choice of \( T \) and \( I \)
Taylor expansion is a good choice of \( T \).

Definition (Function class A) Define \( A_n \) a set of functions \( g : [0,1] \rightarrow R \) in \( C^{m}[0,1] \), such that
\[ g(x) = 1 - x^j, 1 \leq j \leq m. \]

Lemma 0.2 \( g(x) = 1 - x^m \) is in \( A_{m+1} \).

Definition (Function class B) Define \( B \) a set of functions \( g : [0,1] \rightarrow R \), such that
\[ g(x) = 1 - x^m, 1 \leq m \leq n. \]

Definition (Function class C) Define \( C \) a set of functions \( g : [0,1] \rightarrow R \), such that
\[ g(x) = 1 - x, \]

Theorem 0.4 (Candidate 2 of basis 1 function) \( \forall g : [0,1] \rightarrow R \), \( A_n \cap B \) is a basis 1 function of order \( m \), where \( A \) is defined as follow:
\[ H(x) = \frac{1}{1-x} (1-x^2) \]

Theorem 0.5 (Iteration) \( \forall g : [0,1] \rightarrow R \), \( A_n \cap B \) is a basis 1 function of order \( m \), where \( A \) is defined as follow:
\[ \phi(x) = 1 - x \]

Construction of \( I \) function
If \( g \) is a basis 1 function of order \( m \), then we define \( I \) of \( H(x, m) \) as follow
\[ I_1(x) = \begin{cases} 1, & \text{if } x \in [x_1, x_2], \\ 0, & \text{otherwise} \end{cases} \]

Graphs of \( H \) with \( y = 1 - x^m \) and \( m = 1/2 \)

Graphs of \( \phi \), with \( n = 1, 2, 3, 4 \)

Graphs of \( \phi \), with \( n = 1, 2, 3, 4 \)

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Summary

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Smoothness Conditions

Convergence Conditions

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