雪隆八独中电脑工作营
Klang Valley Independent High School Computing Camp

Interesting Problems in Mathematics and Computer Science

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(05 June 2010, Saturday, 12:30—3:30pm)

Experience the fun of problem solving

Interesting Problems in Math-CS

Outline

- ❖ Inverting a Triangle of Coins
- ❖ Sending Information with 0 bit
- ❖ Sending Information with 1 bit
- ❖ Sending Information with 4 bits

Experience the fun of problem solving

First, a Quote.

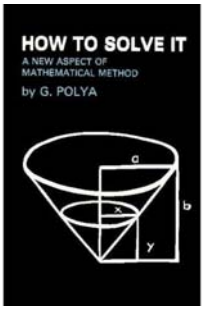
A great discovery solves a great problem,
but there is a grain of discovery
in the solution of any problem.

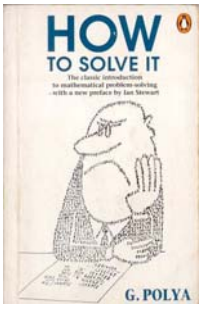
Your problem may be modest;
but if it challenges your curiosity and
bring into play your inventive faculties,
and if you solve it by your own means,
you may experience the tension and
enjoy the triumph of discovery.

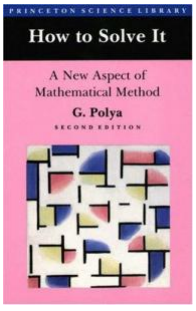
The Aha! moment

G. Polya, 1945

"How to Solve It", George Polya

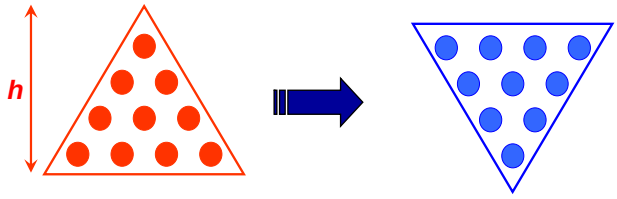






The most widely cited reference
for problem solving in all disciplines

Problem: Inverting a Triangle of Coins

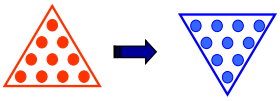


- "Invert" a triangle of coins by moving as few coins as possible
- Let h := "height of triangle"
  m := "# of coins to move"

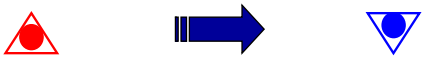
Question: What is the value of m, when $h = 90$?

CHHS celebrated 90th anniversary

Inverting a Triangle of Coins (1)



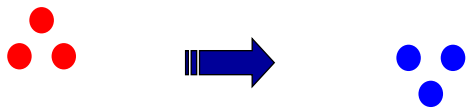
- **Try some small instances…**
  - $h = 1$



- **TRIVIAL:**
  - $h = 1$ $m = 0$

Inverting a Triangle of Coins (2)

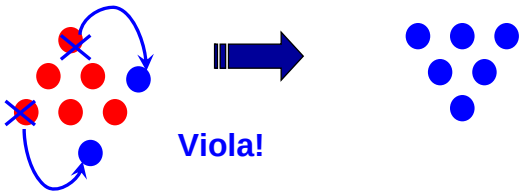
- Now, try something bigger…
  - *h* = 2



- TRIVIAL:
  - *h* = 1 *m* = 0
  - *h* = 2 *m* = 1

Inverting a Triangle of Coins (3)

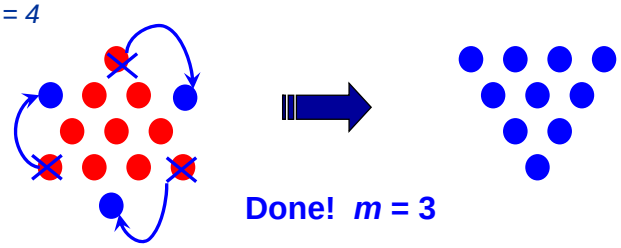
- Now, try something bigger…
  - *h* = 3



- RESULTS:
  - *h* = 1 *m* = 0
  - *h* = 2 *m* = 1
  - *h* = 3 *m* = 2

Inverting a Triangle of Coins (4)

- Now, try original problem…
  - *h* = 4



Done! *m* = 3

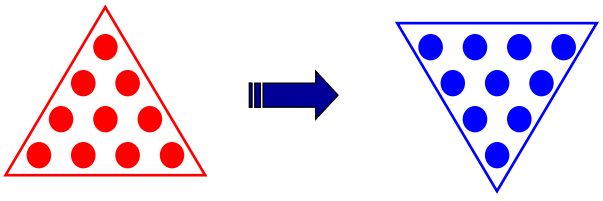
- RESULTS:
  - *h* = 1 *m* = 0
  - *h* = 2 *m* = 1
  - *h* = 3 *m* = 2
  - *h* = 4 *m* = 3

Is there a pattern?

What about *h* = 5? *h* = 90

We Need a Different Perspective!!

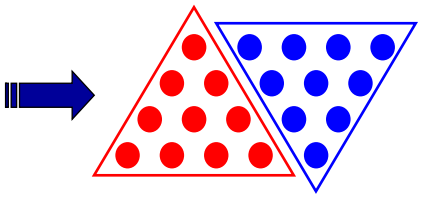
- Let's see….



- How about… seeing them *together?*
- Let's move *THEM*…

We Need a Different Perspective (2)

New Perspective

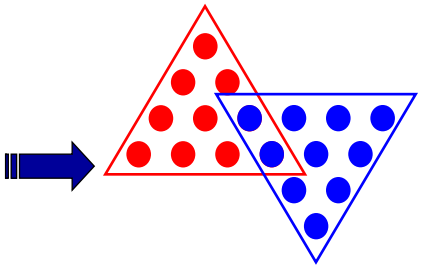


New Question:

How to turn the "orange coins" into the "blue coins"?

Need to move 10 coins!

We Need a Different Perspective (3)

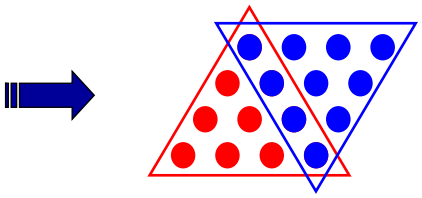


New Operation:

Move the orange triangle around the blue triangle.

Need to move 8 coins!

We Need a Different Perspective (4)

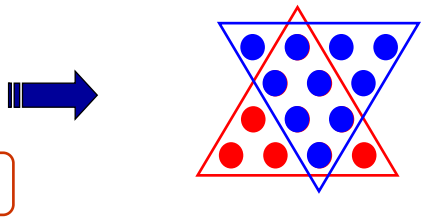


New Operation:

Move the orange triangle around the blue triangle.

Need to move 6 coins!

We Need a Different Perspective (5)

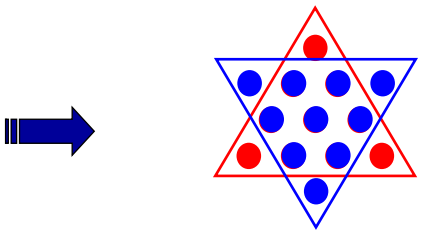


New Insight

Minimize "moves" = Maximize overlap

Need to move 4 coins!

We Need a Different Perspective (6)

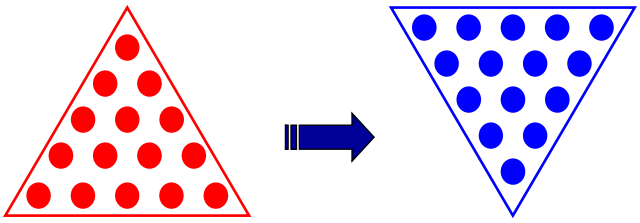


Is this the maximum overlap?

Need to move 3 coins!

Try new insight on larger problem

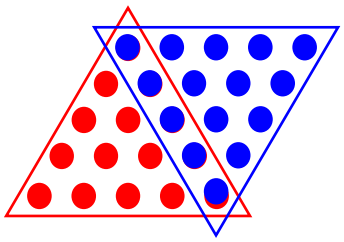
Now try *h* = 5



Move triangle & maximum overlap

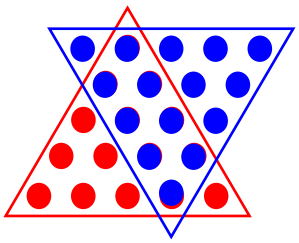
Maximizing Overlap for h=5 (1)

- There are 3 sides in each triangle.
  - *At each side, we have a "non-overlapping" triangle!*
  - *Denote their heights by (a,b,c)*



(0,0,4)

10 coins!

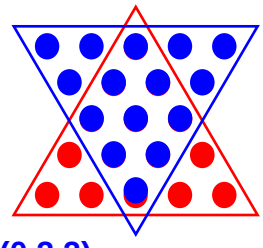


(0,1,3)

7 coins!

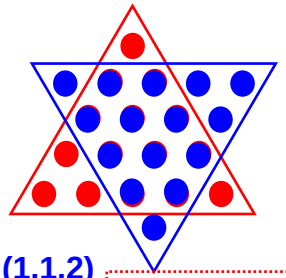
Maximizing Overlap for h=5 (2)

- More configurations…



(0,2,2)

6 coins!



(1,1,2)

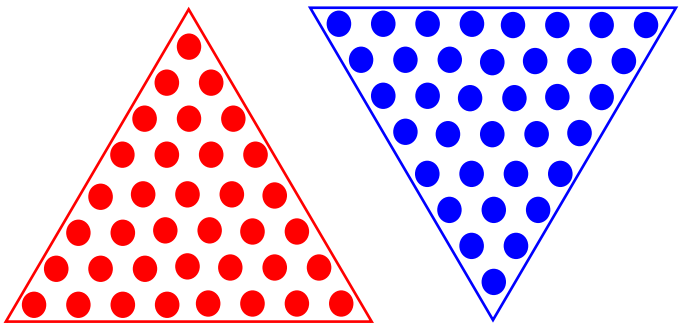
5 coins!

- Note that (a + b + c) = 4
- Which configuration gives minimum *m*?

Activity Period #4:

"Invert Triangle of Coins" for *h* = 8 (DIY)

(5 minutes)

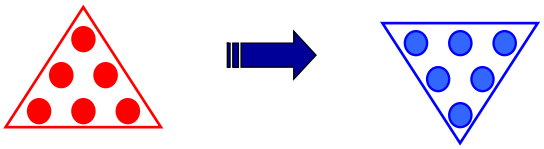


Review of Activity #1

- Did you try with real coins?
  - Yes: _____ No: _____
- What is the value of (a + b + c)?
- What configuration (a, b, c) is *minimum*?
  - (a, b, c) = (__ , __ ,__)
  - How many coins moved? _____ coins
- Was it easy?
- *Was it fun?*

Maximizing Overlap is Easy…

- Try for previous cases h = 3



- "Non-overlapping triangles with hts (a,b,c)
  - *For h=2, we have (0,0,1) m=1*
  - *For h=3, we have (0,1,1) m=2*
  - *For h=4, we have (1,1,1) m=3*
  - *For h=5, we have (1,1,2) m=5*
  - *For h=6, we have (1,2,2) m=7*
  - *For h=7, we have (2,2,2) m=9*
  - *For h=8, we have (2,2,3) m=12*

Looking Back…

- Can work out for h = 9, 10, 11, …, 90
- The ANSWER is NOT so important, the *METHOD IS more important !*
- Where is the *Key Step*? *The Aha! moment*
- Why was it *not apparent* to us *at the beginning* ?
- How *did* we get to this step?

Moral of the Story

- Don't just solve *a* problem.
- Look for a *general* method.
- View problem from *different perspective*.
- Look for an *Aha moment*.
- Ask "How can I solve something similar if I see one in the future?"

Interesting Problems in Math-CS

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Acknowledgements…

❑ Some slides taken/modifed from..

❖ Prof C. L. Liu (Dave)

The Wise Men and the Hats

Game Setup:

- Many wise men in a room, each wearing a hat (white or black)
- Each man cannot see their own hat, but can see that of others

Announcement:

- "In this room, there is at least one wise man with a white hat."
- At every hour, announcer ask "Who is wearing a white hat?".

Prove:

- If there are *k* wise men wearing white hats, then at the k^{th} hour all the *k* wise men will raise their hands.

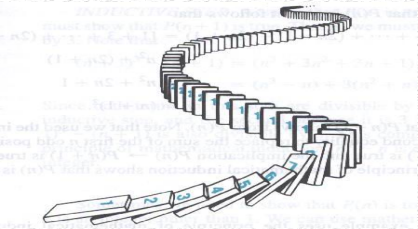
The Wise Men and the Hats

One white hat
1st hour : hand raised

Two white hats
1st hour : silence
2nd hour : hands raised

Five white hats
1st hour : silence
2nd hour : silence
3rd hour : silence
4th hour : silence
5th hour : hands raised

Mathematical Induction



The first domino falls.
If a domino falls,
so will the next domino.

All dominoes will fall !

The Wise Men and the Hats

If there are n wise men wearing white hats, then at the n^{th} hour all the n wise men will raise their hands.

Basis : $n = 1$

At the 1^{st} hour, the only wise man wearing a white hat will raise his hand.

Induction step :

Suppose there are $n+1$ wise men wearing white hats.
At the n^{th} hour, no wise man raises his hand.
At the $n+1^{st}$ hour, all $n+1$ wise men raise their hands.

Interesting Problems in Math-CS

NUS National University of Singapore
School of Computing

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Another Hat Problem

- 10 people in a line, each facing forward
- Each wearing hat (white or black), but don't know which
- Each person can see hat of people *in front of* him
- Each person can hear what others are saying

- Starting from last (leftmost) person in the line, ask "What is the colour of your hat?"
- The person answers only "White" or "Black"
- If answer is correct, man gets to live.
- If answer is incorrect, man is shot!

Another Hat Problem

No strategy

In the worst case, all men were shot.

Strategy 1

In the worst case, half of the men were shot.

Design a strategy so that as few men will die as possible.

Modulo – 2 addition

0 1 2 3 4 5 6 …..

even odd even odd even odd even…..

$\oplus$	even	odd
even	even	odd
odd	odd	even

$\oplus$	+	–
+	+	–
–	–	+

$\oplus$	0	1
0	0	1
1	1	0

Another Hat Problem

0 1 1 0 1

1 ⊕ 1 ⊕ 0 ⊕ ..... ⊕ 1 = 1

1 ⊕ 0 ⊕ ..... ⊕ 1 = 0

1

Another Hat Problem

0 1 1 0 1

1 ⊕ 1 ⊕ 0 ⊕ ..... ⊕ 1 = 1

1 ⊕ 0 ⊕ ..... ⊕ 1 = 0

0 ⊕ ..... ⊕ 1 = 1

1

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Card Trick with 4 "bits" of info

- ❑ You choose any 5 cards from a deck
- ❑ My Asst #1 will show *4 cards* to Asst #2
- ❑ Asst #2 announces the 5th (*hidden*) card

Card Trick (for Audience viewing)

Template: ♦ ♣ ♥ ♠

A 2 3 4 5 6 7 8 9 10 J Q K

- First, volunteer to choose any 5 cards.
- Card shown by Asst #1:

_♥5_ _♦9_ _♦4_ _♠Q_

What's the Secret?

- Coding Theory!
- Homework:
  - Figure out the code for this card trick.
  - Discuss among yourselves
  - Ask team leader for hints
  - But… solve it yourselves
  - Email your answer to me
    - leonghw@comp.nus.edu.sg

Coding Theory

Representation of information in alternate forms for

- efficiency
- reliability
- security

- Algebraic Coding Theory
- Cryptography

References…. for More readings...

- G. Polya, ***"How to Solve It"***,
  - ***Princeton Univ Press (1945),***
  - ***re-published by Penguin Books (1990)***

- W. A. Wickelgren, ***"How to Solve Problems"***,
  - ***W. H. Freeman, 1974.***
- R. R. Kadesch, ***"Problem Solving Across the Disciplines"***,
  - ***Prentice Hall, 1997.***
- Edward De Bono,
  - ***Various Different Titles (from Lateral Thinking, to Teaching Thinking, to Hats, to Simplicity).***

- [EvAn] ***Everywhere and Anywhere -- Use your imagination!***

Thank you!

If you want to contact me,
go email, MSN, FB at leonghw@comp.nus.edu.sg