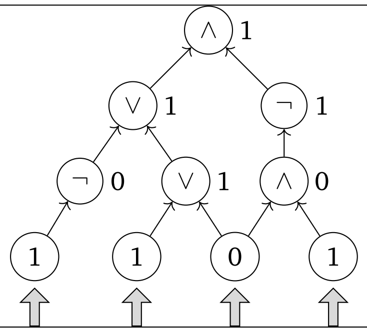
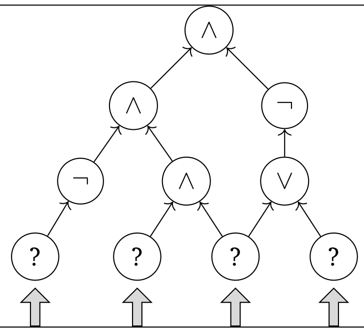
Last Update: – (Initial Version)

List of NP-Complete Problems

These are the NP-Complete problems you will see in the course (lectures/tutorials). Unless otherwise specified, you may assume these problems are NP-Complete in assessments.

You should populate this list with the new NP-Complete problems you see in assignments/problem sets.

- **CIRCUIT-SAT (Circuit Satisfiability)**: Given a directed acyclic graph with nodes corresponding to `AND`, `NOT`, `OR` gates and n binary inputs, does there exist any binary input which gives output 1?

YES-instance	NO-instance
	

> **Theorem:** Every problem from `NP` be reduced to circuit satisfiability.

- **CNF-SAT (Satisfiability)**: Given a CNF formula Φ, does it have a satisfying truth assignment?

> **Definitions:**
> - A **literal** is a Boolean variable or its negation (x_i, $\overline{x_i}$).
> - A **clause** is a disjunction (OR) of literals. (e.g. $C_j = x_1 \vee \overline{x_2} \vee x_3$).
> - A formula is said to be in **Conjunctive Normal Form (CNF)** if it is a conjunction (AND) of clauses (e.g. $\Phi = C_1 \wedge C_2 \wedge C_3 \wedge C_4$).

YES-instance	NO-instance
$(x_1 \vee x_2) \wedge (\overline{x_2} \vee x_3 \vee \overline{x_4}) \wedge (\overline{x_1} \vee x_4)$ Solution: $x_1 = 1, x_2 = 0, x_3 = 0, x_4 = 1$	$x_1 \wedge \overline{x_1}$

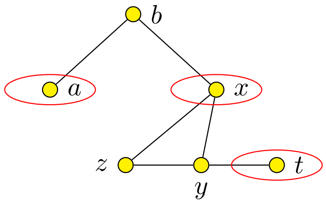
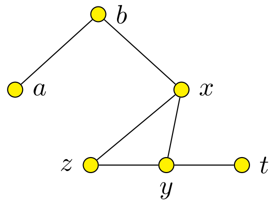
- **3-SAT**: Given a CNF formula Φ where each clause has exactly 3 (not necessarily distinct) literals, does it have a satisfying truth assignment?

YES-instance	NO-instance
$(\overline{x_1} \vee x_2 \vee x_3) \wedge (x_1 \vee \overline{x_2} \vee x_3) \wedge (\overline{x_1} \vee x_2 \vee x_4)$ Solution: $x_1 = 1, x_2 = 1, x_3 = 0, x_4 = 1$	$(x_1 \vee x_1 \vee x_1) \wedge (\overline{x_1} \vee \overline{x_1} \vee \overline{x_1})$

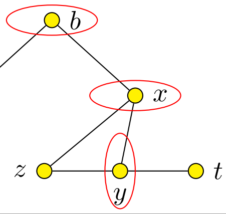
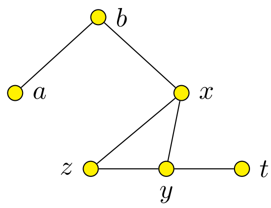
- **POSITIVE-SAT**: Given a CNF formula Φ where each clause only consists of **non-negated** literals corresponding to different variables.
  - Optimization Version: Compute the minimum number of variables that needs to be set true such that the formula is satisfied.
  - Decision Version: Is it possible to satisfy the formula by setting as most k variables to true?

YES-instance	NO-instance
$(x_1 \vee x_2) \wedge (x_1 \vee x_3) \wedge x_4$, $k = 2$ Solution: $x_1 = 1, x_2 = 0, x_3 = 0, x_4 = 1$	$(x_1 \vee x_2) \wedge (x_1 \vee x_3) \wedge x_4$, $k = 1$

- **Independent-Set**: Given an undirected graph $G = (V, E)$, a subset $X \subseteq V$ is said to be an **independent set** if, for each $u, v \in X$, $(u, v) \notin E$.
  - Optimization Version: Compute independent set of the largest size.
  - Decision Version: Does there exist an independent set of size $\geq k$?

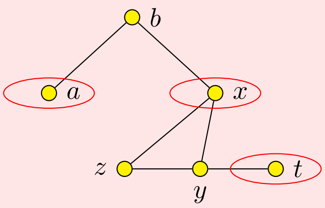
YES-instance	NO-instance
Here shows an independent set of size ≥ 3. 	There are no independent sets of size ≥ 4. 

- **Vertex-Cover**: Given an undirected graph $G = (V, E)$, a subset $X \subseteq V$ is said to be a **vertex cover** if, for each edge $(u, v) \in E$, either $u \in X$ or $v \in X$.
  - Optimization Version: Compute vertex cover of the smallest size.
  - Decision Version: Does there exist a vertex cover of size $\leq k$?

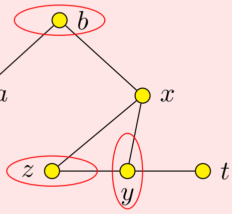
YES-instance	NO-instance
Here shows a vertex cover of size ≤ 3. 	There are no vertex covers of size ≤ 2. 

Theorem: $X \subseteq V$ is a vertex cover of G **if and only if** $V \setminus X$ is an independent set of G.

Example:

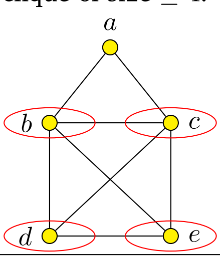
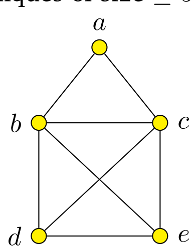


$\longleftrightarrow$

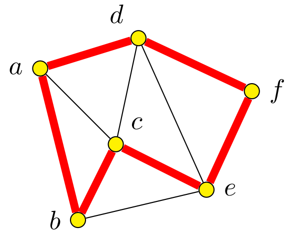
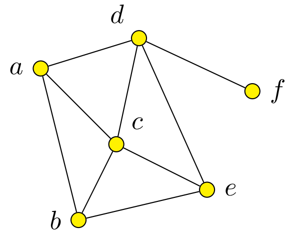


Independent Set *Vertex Cover*

- **Max-Clique**: Given an undirected graph $G = (V, E)$ and an integer k, a **clique** is a subset of the vertex set $C \subseteq V$, such that for every two vertices $u, v \in C$, $(u, v) \in E$ (in other words, the subgraph induced by C is complete).
  - Optimization Version: Compute clique of the largest size.
  - Decision Version: Does there exist a clique of size $\geq k$?

YES-instance	NO-instance
Here shows a clique of size ≥ 4. 	There are no cliques of size ≥ 5. 

- **Hamiltonian-Cycle**: Given an undirected graph $G = (V, E)$, a cycle is said to be **Hamiltonian** if it passes through each vertex exactly once.
  - Optimization Version: Compute the largest Hamiltonian cycle in G.
  - Decision Version: Does there exist a Hamiltonian cycle of size $|V|$?

<table>
<tr><th>YES-instance</th><th>NO-instance</th></tr>
<tr><td>
</td><td>
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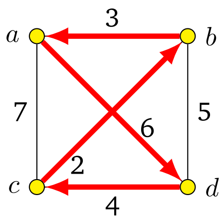
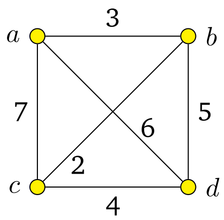
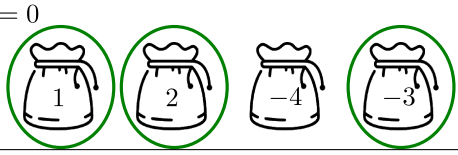
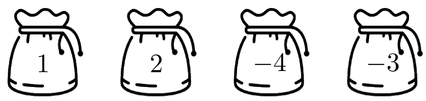
</table>

- **Travelling-Sales-Person**: Given a weighted, undirected, complete graph $G = (V, E)$ with non-negative cost for its edges. The cost of a tour is the sum of the edge costs traversed in the tour.
  - Optimization Version: Compute the minimum cost tour.
  - Decision Version: Does there exist a tour of cost $\leq b$?

> **Definitions:** A tour is a sequence $(v_0, v_1, v_2, \ldots, v_n)$ of vertices in such that:
> - It starts and terminates on the same vertex, i.e. $v_0 = v_n$.
> - There is an edge between every adjacent pair of vertices in the sequence, i.e. for all $1 \leq i \leq n$, $(v_{i-1}, v_i) \in E$.
> - Each vertex in V is visited exactly once, except for v_0 which is visited twice.

<table>
<tr><th>YES-instance</th><th>NO-instance</th></tr>
<tr><td>Here shows a tour with cost ≤ 15.

</td><td>There are no tours with cost ≤ 14.

</td></tr>
</table>

- **Subset-Sum**: Given a set of n non-negative integers $S = \{w_1, \ldots, w_n\}$ and a target W, decide whether there exists a subset $I \subseteq \{1, 2, \ldots, n\}$ such that $\sum_{i \in I} w_i = W$. (In other words, decide whether there exists a subset of S with sum W.)

<table>
<tr><th>YES-instance</th><th>NO-instance</th></tr>
<tr><td>$W = 0$

</td><td>$W = 4$

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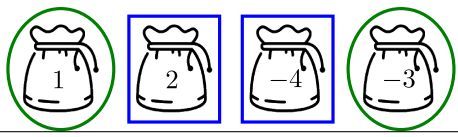
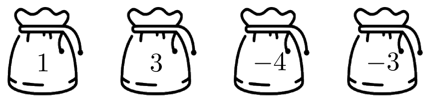
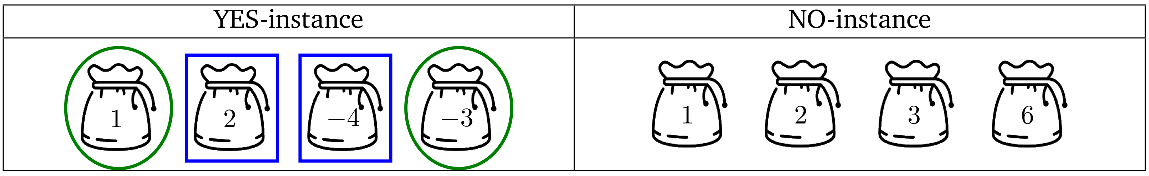
</table>

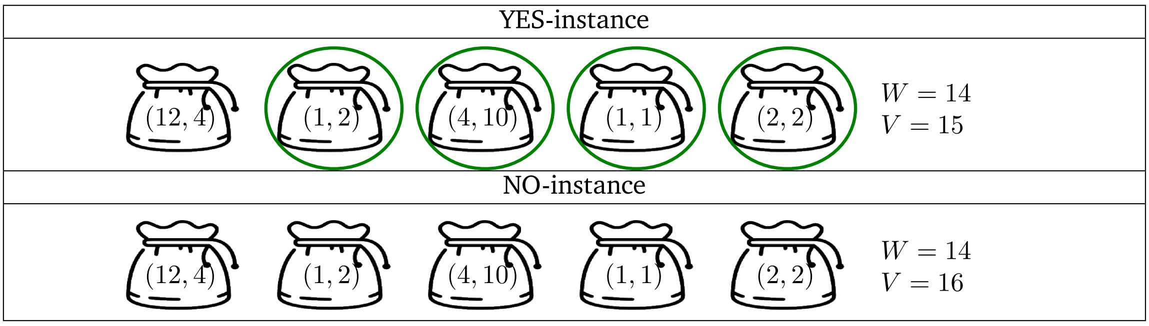
- **Partition**: Given a set of n non-negative integers $S = \{w_1, \ldots, w_n\}$, decide whether $\{1, 2, \ldots, n\}$ can be partitioned into two subsets I_1, I_2 such that $\sum_{i \in I_1} w_i = \sum_{i \in I_2} w_i$. (In other words, decide whether S can be partitioned into two subsets with equal sum.)

<table>
<tr><th>YES-instance</th><th>NO-instance</th></tr>
<tr><td>
</td><td>
</td></tr>
</table>

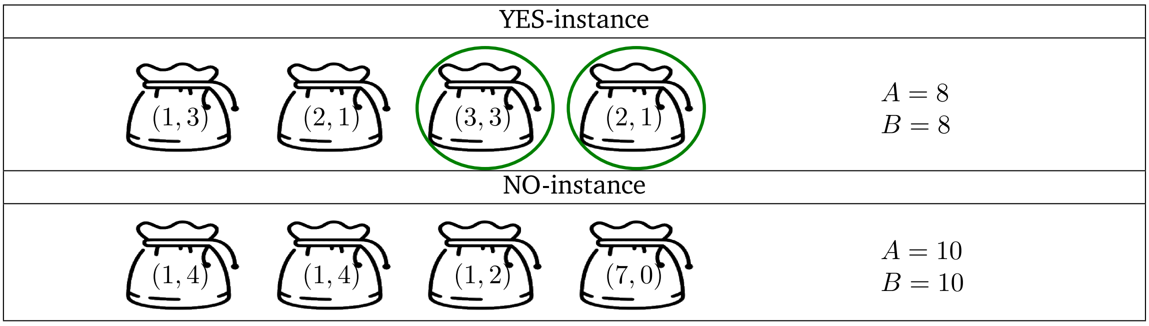
- **PARTITION-SPECIAL**: Given a set of n non-negative integers $S = \{w_1, \ldots, w_n\}$, decide whether $\{1, 2, \ldots, n\}$ can be partitioned into two subsets I_1, I_2 such that $\sum_{i \in I_1} w_i = \sum_{i \in I_2} w_i$ and $|I_1| = |I_2|$. (In other words, decide whether S can be partitioned into two subsets with equal sum **and equal size.**)



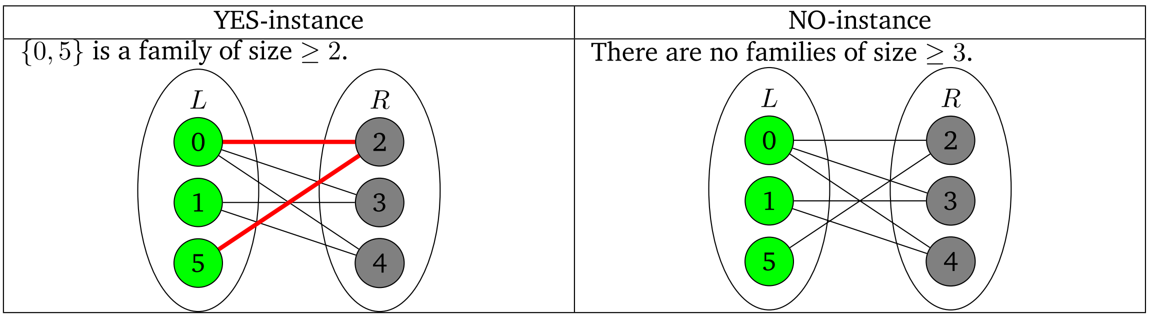
- **KNAPSACK**: Given a set of n items with weight, value pairs $(w_1, v_1), (w_2, v_2), \ldots, (w_n, v_n)$ and a capacity W, compute a subset $I \subseteq \{1, 2, \ldots, n\}$ such that $\sum_{i \in I} w_i \leq W$.
  - Optimization Version: Compute the subset that maximizes $\sum_{i \in I} v_i$.
  - Decision Version: Does there exist a subset such that $\sum_{i \in I} v_i \geq V$?



- **FANTASTIC-HALF**: Given a set of n items with value pairs $(a_1, b_1), (a_2, b_2), \ldots, (a_n, b_n)$. Let $A = \sum_i a_i$ and $B = \sum_i b_i$. Does there exist a subset $I \subseteq \{1, 2, \ldots, n\}$ of size $n/2$ such that $\sum_{i \in I} a_i \geq A/2$ and $\sum_{i \in I} b_i \geq B/2$?



- **FIND-FAMILY**: An undirected bipartite graph $G = (L \cup R, E)$ has two disjoint vertex sets L and R where each edge has one endpoint in L and another in R. We call a pair $u, v \in L$ **siblings** if there exists a vertex in $r \in R$ such that both the edges (u, r) and (r, v) are present. A subset $F \subseteq L$ is said to be a **family** if for all distinct $u, v \in F$, u and v are siblings.
  - Optimization Version: Find the family with the maximum size.
  - Decision Version: Does there exist a family of size $\geq k$?



- **SET-COVER**: Given a collection of sets $\{S_1, S_2, \dots, S_n\}$, where each set is a subset of $\{1, 2, \dots, n\}$. A sub-collection of sets is a **set cover** if their union is equal to $\{1, 2, \dots, n\}$.
  - Optimization Version: Compute the set cover with the smallest number of sets.
  - Decision Version: Does there exist a set cover with $\le k$ sets?

YES-instance	
$S_1 = \{$ 1 2 3 $\}$	$k = 3$
$S_2 = \{$ 3 4 $\}$	
$S_3 = \{$ 1 2 4 5 $\}$	
$S_4 = \{$ 1 5 6 $\}$	
NO-instance	
S_1, S_2, S_3, S_4 as given above.	$k = 2$