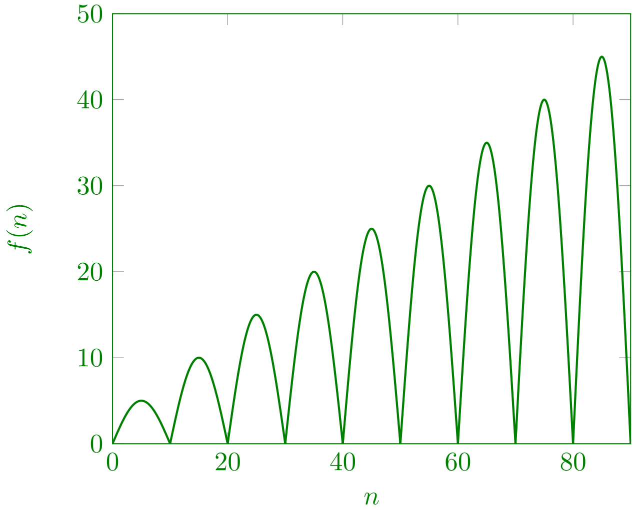
1. True or False: Suppose $f(n), g(n) \geq 1$. If $2^{f(n)} \in O(2^{g(n)})$, then $f(n) \in O(g(n))$.

   Answer: True
   By definition, $\exists c \; \exists n_0 \; \forall n \geq n_0 \; 2^{f(n)} \leq c \cdot 2^{g(n)}$.
   Taking log on both sides, $\exists c \; \exists n_0 \; \forall n \geq n_0 \; f(n) \leq \log c + g(n) \leq \log c \cdot g(n)$.
   Remark. This does not work if $f(n)$ and $g(n)$ tends to 0 when $n \to \infty$.

2. True or False: Suppose $f(n), g(n) \geq 1$. If $f(n) \notin O(g(n))$, then $f(n) \in \Omega(g(n))$.

   Answer: False
   Consider the following function f. $f(n)$ is neither in $O(1)$ nor $\Omega(1)$.



3. Solve $T(n) = T(\frac{n}{2}) + T(\frac{n}{3}) + n$.
   Answer: $T(n) = \Theta(n)$
   Substituting $T(n) = n$, we get $\frac{5n}{6} < n$ for the recursive part and $n = \Theta(n)$ for the merging part. This means our guess is correct.
   Proof Sketch.
   Lower bound: $T(n) \geq n$.
   Upper bound (induction step): $T(n) = T(\frac{n}{2}) + T(\frac{n}{3}) + n \leq c(\frac{n}{2}) + c(\frac{n}{3}) + n \leq cn$, by picking $c \geq 6$.

4. Solve $T(n) = T(\frac{n}{2}) + T(\frac{n}{3}) + T(\frac{n}{6}) + n$.
   Answer: $T(n) = \Theta(n \log n)$
   Substituting $T(n) = n$, we get $n = n$ for the recursive part and $n = \Theta(n)$ for the merging part. This falls into Case #2 of Master Theorem where every level has equal work, so we have to multiply the answer by $\log n$.
   Proof Sketch.
   Lower bound (induction step): $T(n) \geq c_1(\frac{n}{2}\log\frac{n}{2} + \frac{n}{3}\log\frac{n}{3} + \frac{n}{6}\log\frac{n}{6}) + n \geq c_1 n \log\frac{n}{6} + n = c_1 n \log n - c_1 n \log 6 + n \geq c_1 n \log n$, by picking $c_1 \leq \frac{1}{\log 6}$.
   Upper bound (induction step): $T(n) \leq c_2(\frac{n}{2}\log\frac{n}{2} + \frac{n}{3}\log\frac{n}{3} + \frac{n}{6}\log\frac{n}{6}) + n \leq c_2 n \log\frac{n}{2} + n = c_2 n \log n - c_2 n \log 2 + n \leq c_2 n \log n$, by picking $c_2 \geq \frac{1}{\log 2}$.

5. Solve $T(n) = 2T(\frac{n}{2}) + 8T(\frac{n}{4}) + n$.
   Answer: $T(n) = \Theta(n^2)$
   Substituting $T(n) = n^2$, we get $\frac{n^2}{2} + \frac{n^2}{2} = n^2$ for the recursive part and $n = o(n^2)$ for the merging part. This means our guess is correct.
   Proof Sketch.
   Lower bound (induction step): $T(n) \geq 2c_1(\frac{n}{2})^2 + 8c_1(\frac{n}{4})^2 + n \geq c_1 n^2 + n \geq c_1 n^2$.
   Upper bound (induction step): $T(n) \leq 2\left(c_2(\frac{n}{2})^2 - c_3\frac{n}{2}\right) + 8\left(c_2(\frac{n}{4})^2 - c_3\frac{n}{4}\right) + n = c_2 n^2 - 3c_3 n + n \leq c_2 n^2 - c_3 n$, by picking $c_3 \geq 0.5$.

6. Solve $T(n) = 2T(\frac{n}{2}) + 3T(\frac{n}{3}) + n^2$.

   Answer: $T(n) = \Theta(n^2)$
   Substituting $T(n) = n^2$, we get $\frac{n^2}{2} + \frac{n^2}{3} < n^2$ for the recursive part and $n^2 = \Theta(n^2)$ for the merging part. This means our guess is correct.
   Proof Sketch.
   Lower bound: $T(n) \geq n^2$.
   Upper bound (induction step): $T(n) \leq 2c(\frac{n}{2})^2 + 3c(\frac{n}{3})^2 + n^2 = \frac{5}{6}cn^2 + n^2 \leq cn^2$, by picking $c \geq 6$.

7. Solve $T(n) = T(\frac{n}{2}) + 0.5T(n-1) + n$.

   Answer: $T(n) = \Theta(n \log n)$
   Substituting $T(n) = n$, we get $\frac{n}{2} + \frac{n}{3} = n$ for the recursive part and $n = \Theta(n)$ for the merging part. This falls into Case #2 of Master Theorem where every level has equal work, so we have to multiply the answer by $\log n$.
   Proof Sketch.
   Lower bound (induction step): $T(n) \geq c_1 \frac{n}{2} \log \frac{n}{2} + 0.5c_1(n-1)\log(n-1) + n \geq c_1(n-1)\log \frac{n}{2} + n \geq c_1 n \log n$, by picking $c_1 \leq \frac{1}{\log 2+1}$.
   Upper bound (induction step): $T(n) \leq c_2 \frac{n}{2} \log \frac{n}{2} + 0.5c_2(n-1)\log(n-1) + n \leq c_2 \frac{n}{2}(\log n - 1) + 0.5c_2 n \log n + n = c_2 n \log n - c_2 \frac{n}{2} + n \leq c_2 n \log n$, by picking $c_2 \geq 2$.

8. Solve $T(n) = 2\sqrt{T(n-1)} + n$.

   Answer: $T(n) = \Theta(n)$
   Substituting $T(n) = n$, we get $2\sqrt{n-1} < n$ for the recursive part and $n = \Theta(n)$ for the merging part. This means our guess is correct.
   Proof Sketch.
   Lower bound: $T(n) \geq n$.
   Upper bound (induction step): $T(n) \leq 2\sqrt{c_2(n-1)} + n \leq 2\sqrt{c_2}n + n \leq c_2 n$, by picking $c_2 \geq 9$.