National University of Singapore

School of Computing

CS3230 - Design and Analysis of Algorithms

Midterm Test

(Semester 2 AY2024/25)

Time Allowed: 80 minutes

INSTRUCTIONS TO CANDIDATES:

1. **DO NOT** open this assessment paper until instructed to do so.
2. This paper has **2 sections** across **12 printed pages**, including the cover page. Page 12 is blank and can be used for scratch work; any writing on that page will **NOT** be graded.
3. This is an **Open Book** assessment; however, the use of calculators or other electronic devices is **not allowed**.
4. Write and shade your **Student Number** using a 2B pencil in the box on the **The Answer Sheet** (page 5).
5. For Section A (MCQ questions), fill in the bubbles on page 5 using a 2B pencil. No marks will be awarded for a blank answer.
6. For Section B, answer **ALL** questions within the **provided boxed spaces**.
   - Leaving a blank answer for a question will result in 1 mark being awarded.
   - However, if you write even a single character, and your answer is entirely incorrect, it will result in 0 marks.
   - You may use either a pen or a pencil; however, ensure that your writing is **legible**.
7. Important tips:
   - Pace yourself and avoid spending too much time on a single question.
   - Read all questions carefully before starting; some may be simpler than they appear.
8. Unless otherwise specified, all logarithms are assumed to be in base 2.
9. This paper is worth a total of **45 marks**, which will be scaled to **30%** of the overall grade. The marks allocated for each question are indicated in the right margin.

A Multiple Choice Questions (15 marks)

A.1. $n^{10} - n^9$ is in (1½)

A. $\Omega(n^{11})$

B. $o(n^{10})$

C. $\Theta(n^9)$

D. $O(n^8)$

E. None of the above

A.2. $(n+1)!$ is in (1½)

A. $O(n!)$

B. $\omega(n!)$

C. $\Theta(n!)$

D. $o(n!)$

E. None of the above

A.3. $2^{\log_3 n}$ is in (1½)

A. $O(\log_2 n)$

B. $\Theta(n^2)$

C. $\omega(n)$

D. $\Omega(\sqrt{n})$

E. None of the above

A.4. Suppose $f(n) \in \Theta(n^2(\log n)^5)$ and $g(n) \in \Theta(n^5(\log n)^2)$. Then, $f(n) + g(n)$ is in (1½)

A. $\Theta(n^5(\log n)^5)$

B. $\Theta(n^2(\log n)^5)$

C. $\Theta(n^5(\log n)^2)$

D. $\Theta(n^7(\log n)^7)$

E. None of the above

For questions A.5–A.7, you may assume that $T(n) \in \Theta(1)$ for $n \leq 100$.

A.5. Suppose $T(n) = 36T(n/6) + 2n + n^{8/3}$. Then, $T(n)$ is in (1½)

A. $\Theta(n^{8/3})$

B. $\Theta(n^{8/3}\log n)$

C. $\Theta(n^2)$

D. $\Theta(n^2 \log n)$

E. None of the above

A.6. Suppose $T(n) = 64T(n/4) + 3n^{1.5}$. Then, $T(n)$ is in (1½)

A. $\Theta(n^2)$

B. $\Theta(n^3)$

C. $\Theta(n^{1.5})$

D. $\Theta(n^{1.5} \log n)$

E. None of the above

A.7. Suppose $T(n) = T(n/5) + 2T(n/3) + n$. Then, $T(n)$ is in (1½)

A. $\Theta(n)$

B. $\omega(n^2)$

C. $\Omega(n \log n)$

D. $o(n)$

E. None of the above

A.8. For any randomized algorithm, let $E(n)$ and $T(n)$ denote the expected and worst-case running time, respectively, for inputs of length n. Then, which of the following statement is always **TRUE**, irrespective of the randomized algorithm being considered? (1½)

A. For every n, $E(n) < T(n)$

B. For every n, $E(n) = T(n)$

C. For every n, $E(n) > T(n)$

D. For at least one n, $E(n) < T(n)$, and for at least one n, $E(n) > T(n)$

E. None of the above

A.9. Suppose we throw 3 balls independently and uniformly at random into 5 bins. Then, (1½)

A. The probability that all the balls fall into the same bin is 0.

B. The probability that all the balls fall into the same bin is $\frac{3}{5}$.

C. The probability that all the balls fall into the same bin is $\frac{1}{25}$.

D. The probability that all the balls fall into the same bin is $\frac{1}{9}$.

E. None of the above.

A.10. Consider an undirected graph $G = (V, E)$ with $n = |V|$ vertices and $m = |E|$ edges. A randomized algorithm selects a vertex $v \in V$ uniformly at random and returns $\deg(v)$, where $\deg(v)$ denotes the degree of vertex v. Let X be the random variable that denotes the output of this algorithm. What is the expected value of X, i.e., $\mathbb{E}[X]$? (1½)

A. m

B. n

C. m/n

D. $2m/n$

E. None of the above

B Essay Questions (30 marks)

B.1. Consider the following variant of Tower of Hanoi. There are three rods and $n \geq 1$ disks of different diameters. Initially, all disks are stacked on the first rod in increasing order of size from top to bottom (i.e., the smallest disk is at the top, and the largest disk is at the bottom). Alice's task is to move all n disks to the third rod while obeying the following rules.

(i) Only one disk can be moved at a time, and it must be the top disk on some rod.

(ii) No disk can be placed on top of a disk that is smaller than it.

(iii) Alice can only move a disk from a rod to an **adjacent** rod (i.e., from the first rod to the second rod; from the second rod to the first or third rod; and from the third rod to the second rod).

Let $f(n)$ be the number of moves that Alice needs in order to accomplish this task. (You should try to make the number of moves as small as possible for Alice, but you do not need to prove that your number is optimal.)

(a) Write a recurrence for $f(n)$, including the base case(s), and explain how you derived it. (5)

(b) Solve the recurrence from part (a) (i.e., give a closed-form formula for $f(n)$, with justification). (5)

B.2. In teacher Bob's class, there are 10 students. Bob wants to distribute 20 candies among these students. Each student specifies a positive integer value for each candy in such a way that the student's 20 values are all distinct and sum up to 3230. After seeing all the students' values, Bob is allowed to choose an ordering of the 10 students. Then, the students will pick the candies in two rounds, where in each round they queue up to pick one after another according to the ordering chosen by Bob. At each student's turn, the student will pick her highest-value candy among the remaining ones. Hence, each student will end up with 2 candies, and her *final value* is the sum of her values for these 2 candies. (10)

Prove that Bob can always choose an ordering such that the sum of all 10 students' final values is at least 3230.

B.3. There are 100 coins. Charlie knows that 4 of them are fake, but he does not know which ones. He knows that every real coin has the same weight and every fake coin has the same weight (but he does not know these weights), and that each fake coin is lighter than each real coin. He has a balance, which allows him to check for any two (disjoint) sets of coins A and B which of the following three possibilities is true: (10)

(i) A is heavier than B;

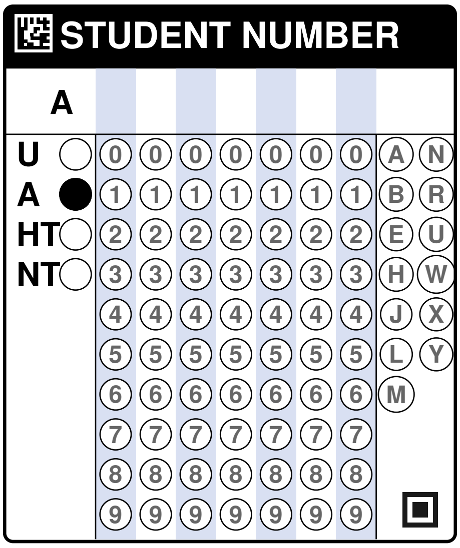
(ii) A is lighter than B;

(iii) A and B weigh equally.

Determine, with proof, a small number k such that by using at most k weighings, Charlie can always point to one coin and say with certainty that this coin is real. (You should try to make k as small as possible for Charlie, but you do not need to show that your k is optimal.)

The Answer Sheet

Using a **2B pencil**, write your **Student Number** and shade the corresponding bubbles. **DO NOT write your name**.



Write your MCQ answers in the <u>special MCQ answer box below</u> for automatic grading. Answers will not be manually checked. Use a **2B pencil**, shade **one** circle per question fully and clearly, and erase any changes completely. Do not cross out answers.

Q	A.1	A.2	A.3	A.4	A.5	A.6	A.7	A.8	A.9	A.10
A	○	○	○	○	○	○	○	○	○	○
B	○	○	○	○	○	○	○	○	○	○
C	○	○	○	○	○	○	○	○	○	○
D	○	○	○	○	○	○	○	○	○	○
E	○	○	○	○	○	○	○	○	○	○

Q B.1.a. Write a recurrence for $f(n)$, including the base case(s), and explain how you derived it.

(Leaving a blank answer will result in 1 mark being awarded.)

Q B.1.b. Solve the recurrence from part (a) (i.e., give a closed-form formula for $f(n)$, with justification).

(Leaving a blank answer will result in 1 mark being awarded.)

Q B.2. Prove that Bob can always choose an ordering such that the sum of all 10 students' final values is at least 3230.

(Leaving a blank answer will result in 1 mark being awarded.)

Q B.3. Determine, with proof, a small number k such that by using at most k weighings, Charlie can always point to one coin and say with certainty that this coin is real.

(Leaving a blank answer will result in 1 mark being awarded.)

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