National University of Singapore

School of Computing

CS3230 - Design and Analysis of Algorithms

Midterm Test

(Semester 1 AY2025/26)

Time Allowed: 120 minutes

INSTRUCTIONS TO CANDIDATES:

1. Do **NOT** open this assessment paper until you are told to do so.
2. This assessment paper contains TWO (2) sections.
   It comprises NINETEEN (19) printed pages, including this page.
3. This is an **Open Book** Assessment.
   You cannot use any electronic device except one non-programmable calculator.
4. For Section A, use the boxes on page 13 (use 2B pencil).
   For Section B, answer **ALL** questions within the **boxed space**.
   If you leave the boxed space blank, you will get automatic free 0.5 mark for that box.
   However, if you write at least a single character and it is totally wrong, you will get 0 mark.
   Canceling your (incomplete) answer can be done by crossing the entire box with a big **X**.
   You can use either pen or pencil. Just make sure that you write **legibly**!
5. Important tips: Pace yourself! Do **not** spend too much time on one (hard) question.
   Read all the questions first! Some questions might be easier than they appear.
6. You can assume that all **logarithms are in base** 2.
7. The total marks of this paper is 60 marks.
   It will then be scaled to 30% of the course weightage.

This page is intentionally left blank

A MCQs ($20 \times 1.5 = 30$ marks)

1. Which of the following statements is **TRUE**?

   a). $n \log n \in \Theta(n)$ clearly not

   b). $2^n \cdot \sqrt{n} \in \Theta(2^n)$ clearly not

   c). $(n+1)^n \in \Theta(n^n)$ Answer, rated 70-30 MCQ

   d). $101^{\log \log n} \in \Theta(100^{\log \log n})$ the base are different, 101^x vs 100^x for the same x, the 101^x grows faster

   e). None of the above c). is TRUE

2. Which of the following functions grows the fastest, asymptotically, as n increases?

   a) $n \log n$

   b) $2^{\log n}$

   c) $n^{\log n}$

   d) $(\log n)^n$ Answer; the power is n, the base is a growing function, rated 80-20 MCQ

   e) 2^n close to option d), but the base is constant

3. You are asked to solve a difficult computation problem in $o(n \log^2 n)$.
   You have been shown a proof that the lower bound to solve this problem is $\Omega(n \log^{\frac{2}{3}} n)$.
   Which target time complexity should we should aim for?

   a). $\Theta(n \log^{\frac{1}{2}} n)$ Impossible, as it breaks the proven lower bound.

   b). $O(n \log^2 n)$ Big O is not little-$o(n \log^2 n)$

   c). $O(\log n)$ Impossible, as it breaks the proven lower bound.

   d). $O(n \log^{\frac{7}{4}} n)$ Yes, fits the requirement, rated 80-20 MCQ

   e). $\Theta(n^2)$ Not improving, as this is above $n \log^2 n$

4. What is the *best (clearest)* way to show that $n \log n \in o(n \log^2 n)$?

   a). For *any* constant $c > 0$ and $n_0 = 7$, for all $n \geq n_0 : 0 \leq n \log n < c \cdot n \log^2 n$ c can be a small fraction and n must be MUCH bigger

   b). $\lim_{n \to \infty} \frac{n \log n}{n \log^2 n} = 7$ No, the limit is wrong (and this is to try showing Θ, not o)

   c). $\lim_{n \to \infty} \frac{n \log^2 n}{n \log n} = \infty$ Yes, this shows ω, then we use complementarity property, rated 60-40 MCQ

   d). For some $k \geq 0$, $n \log n \in \Theta(n \log^{k=1} n)$, so $n \log n \in o(n \log^{k=2} n)$ This is not master theorem case 2; under the latest wording of pick the 'best (clearest)' option, this should NOT be selected as there is a much better option c).

   e). None of the above Option c). is correct

5. Consider the recurrence relation: $T(n) = T(\sqrt{n}) + \sqrt{\log n}$.
   Which of the following statements is **TRUE**?

   a) $T(n) \in \Theta(\log \log n)$

   b) $T(n) \in \Theta(\sqrt{\log n})$ Answer; try drawing the recursion tree; root does the most work, rated 70-30 MCQ

   c) $T(n) \in \Theta(\sqrt{\log n \log \log n})$

   d) $T(n) \in \Theta(\sqrt{\log n} \log \log n)$

   e) None of the above

6. Which of the following is a solution to the recurrence $T(n) = 9 \cdot T(n/3) + n^3$?

   a) $\Theta(n^2)$

   b) $\Theta(n^2 \log n)$

   c) $\Theta(n^3)$ Answer; simple MT application, case 3; no need to actually do regularity check as the format is very standard, rated 90-10 MCQ

   d) $\Theta(n^3 \log n)$

   e) None of the above

7. Consider the following algorithm:
   ALG1($A[1 \dots n]$):

   - For $i = 1$ to $n - 1$:
     - For $j = 1$ to $i - 1$:
       - If $A[j] > A[i]$: swap the values of $A[j]$ and $A[i]$

   Which of the following is a loop invariant that is true at the start of iteration i of the outer loop?

   a) A is sorted

   b) $A[1 \dots (i-1)]$ is sorted Answer, rated 40-60 MCQ

   c) ($A[1 \dots (i-1)]$ is sorted) and ($x \leq A[i]$ for all $x \in A[1 \dots (i-1)]$)

   d) ($A[1 \dots (i-1)]$ is sorted) and ($x \leq y$ for all $x \in A[1 \dots (i-1)]$ and $y \in A[i \dots n]$)

   e) None of the above

8. You are given the Single-Source Shortest Paths (SSSP) problem on a directed weighted graph $G = (V, E)$ where $n = |V|$ and $m = |E|$.
   You are told that G is a *planar graph* where $m \in O(n)$.
   Which algorithm among the five options below has the fastest asymptotic running time?

   a). Dijkstra's algorithm with Binary heap $O((n+n)\log n) \in O(n \log n$

   b). Dijkstra's algorithm with Fibonacci heap Also $O(n + n \log n) \in O(n \log n)$

   c). Use the Duan et al. algorithm (STOC 2025) with time complexity $O(m \log^{\frac{2}{3}} n)$ $O(n \log^{\frac{2}{3}} n)$, correct and the fastest among the five options, rated 60-40 MCQ

d). BFS Wrong algorithm

e). DFS Wrong algorithm

9. Recall that BINARYSEARCH$(A, \text{lb}, \text{ub}, x)$ works as follows.

   - If $\text{lb} > \text{ub}$, return **NO**.
   - Else
     - $\text{mid} \leftarrow \lfloor (\text{lb} + \text{ub})/2 \rfloor$.
     - If $x = A[\text{mid}]$, return **YES**.
     - If $x > A[\text{mid}]$, return BINARYSEARCH$(A, \text{mid}+1, \text{ub}, x)$.
     - If $x < A[\text{mid}]$, return BINARYSEARCH$(A, \text{lb}, \text{mid}-1, x)$.

   If we replace $\text{mid} \leftarrow \lfloor (\text{lb} + \text{ub})/2 \rfloor$ with $\text{mid} \leftarrow \lfloor \text{lb} + \sqrt{\text{ub} - \text{lb}} \rfloor$, then what is the resulting worst-case time complexity for searching in a sorted array of size n?

   a). $\Theta(1)$

   b). $\Theta(\log n)$

   c). $\Theta(\sqrt{n})$ Answer; interesting 'Binary' (Sqrt?) Search variant, $T(n) = T(n - \sqrt{n}) + \Theta(1)$, rated 50-50 MCQ

   d). $\Theta(n)$ Some will pick this, thinking this is similar to the $T(n) = T(n-1) + \Theta(1)$ analysis

   e). None of the above

10. Consider the problem of finding the maximum sum of any sub-array of a given array of integers of size n. (A sub-array is a set of contiguous elements of the given array.) One approach to do this is to divide the array into two equal halves, find the solution for each of them, and then compare these to the maximum sum of a sub-array that starts in the first half and ends in the second half. Assuming the last part is computed with a linear-time algorithm, what is the complexity of the entire algorithm?

    a) $\Theta(n)$

    b) $\Theta(n \log n)$ Answer; kinda similar with Merge Sort analysis, rated 80-20 MCQ

    c) $\Theta(n^2)$

    d) $\Theta(n^2 \log n)$

    e) None of the above

11. If Quicksort always chooses the first element as the pivot and the input array is already sorted, then what is the number of comparisons?

    a). $\Theta(n)$

    b). $\Theta(n \log n)$

    c). $\Theta(n^2)$ Answer, some students may have memorized the answer, rated 90-10 MCQ

    d). $\Theta(n^2 \log n)$

e). None of the above

12. Which of the five events below has the *highest* probability of happening?

    a). Randomly picking an integer from set $\{1, 2, 3, 4, 5, 6, 7\}$ and it is odd $p = \frac{4}{7} = 0.57$ (we allow calculator)

    b). Randomly picking an integer from set $\{2, 3, 4, 5\}$ and it is a prime $p = \frac{3}{4} = 0.75$

    c). Randomly picking an element from an array of size n that contains a majority element appearing at most $\lfloor \frac{3 \cdot n}{5} \rfloor$ times, and the picked element is the majority element

    d). Randomly picking $x \in [-1.0..1.0]$ and $y \in [-1.0..1.0]$ and found that (x, y) is inside a circle centered at $(0, 0)$ with radius $r = 1.0$ $p = \frac{\pi}{4} = \frac{3.1415...}{4} = 0.78$ (we allow calculator, anyway, this is one of their PA1 task, rated 80-20 MCQ)

    e). Randomly picking $x \in [0.0..1.0]$ and $x \geq 0.3$ $p = \frac{1-0.3}{1} = 0.70$

13. Consider the following algorithm:
    ALG2(a, b):

    - Set $s = 0$
    - For $i = a$ to b: $s = s + i$
    - Output s

    Determine the average-case complexity of the above algorithm with its inputs a and b are chosen in the following manner: a is chosen to be a random number from $\{1, 2, \ldots, n\}$, and then b is chosen to be a random number from $\{a, a + 1, \ldots, n\}$.

    a) $\Theta(n)$ Answer, rated 70-30 MCQ

    b) $\Theta(n \log n)$

    c) $\Theta(n^2)$

    d) $\Theta(n^2 \log n)$

    e) None of the above

14. Partition the vertex set of a graph $G = (V, E)$ (without self-loops) into two parts $V = V_1 \cup V_2$ randomly as follows.

    - Each vertex $v \in V$ flips a coin independently.
      - If the outcome is head, v joins V_1.
      - If the outcome is tail, v joins V_2.

    Suppose the coin is biased in such a way that the the outcome is head with probability $2/3$ and is tail with probability $1/3$. What is the expected number of edges crossing V_1 and V_2?

    a). $(3/9)|E|$ Very close option for those who are not careful, thinking it is $\frac{1}{3}|E|$

    b). $(4/9)|E|$ Answer, $Pr(u \in V_1, v \in V_2) + Pr(u \in V_2, v \in V_1) = \frac{2}{3} \cdot \frac{1}{3} + \frac{1}{3} \cdot \frac{2}{3} = \frac{4}{9}$, rated 50-50 MCQ

c). $(5/9)|E|$

d). $(6/9)|E|$

e). None of the above

15. Consider the following randomly generated graph with n vertices $V = \{1, 2, \ldots, n\}$ (n is a multiple of 3): For each pair of vertices i and j (where $i \neq j$), the edge $\{i, j\}$ exists in the graph with probability 1/3. What is the expected number of undirected edges that go between vertices in the set $\{1, 2, \ldots, n/3\}$ and those in the set $\{n/3 + 1, n/3 + 2, \ldots, n\}$?

(a) $2n^2/9$ some students will pick this

(b) $2n^2/27$ Answer, there are at most $\frac{n}{3} \cdot \frac{2n}{3} = \frac{2n^2}{9}$ edges in bipartite graph constructed this way, but since the probability of edge appearing is $\frac{1}{3}$, overall is $\frac{2n^2}{27}$, rated 60-40 MCQ

(c) $n^2/9$

(d) $n^2/27$

(e) None of the above

16. You can climb a staircase with n steps by jumping either one or two steps at a time. Let $T(n)$ be the number of distinct ways to climb n steps. For example, $T(1) = 1$ (only one way: 1), $T(2) = 2$ (two ways: 1+1, 2), and $T(3) = 3$ (three ways: 1+1+1, 1+2, 2+1). What is the value of $T(10)$?

a). 23

b). 45

c). 55 This is $T(9) = Fib(10) = 55$

d). 89 Answer; $T(10) = Fib(11) = 89$, as shown in first lecture, LeetCode demo: climbing-stairs, rated 80-20 MCQ

e). None of the above

17. Suppose the correct 7-lines (Python) implementation shown in class for `https://leetcode.com/problems/longest-common-subsequence`:

```
class Solution:
  def longestCommonSubsequence(self, text1: str, text2: str) -> int:
    @cache
    def LCS(i, j):
      if i < 0 or j < 0: return 0
      return 1+LCS(i-1,j-1) if text1[i]==text2[j] else max(LCS(i-1,j), LCS(i,j-1))
    return LCS(len(text1)-1, len(text2)-1)
```

is modified (only the last 3 lines are changed) into:

```
class Solution:
  def longestCommonSubsequence(self, text1: str, text2: str) -> int:
    @cache
    def LCS(i, j):
      if i == len(text1) or j == len(text2): return 0
      return 1+LCS(i+1,j+1) if text1[i]==text2[j] else max(LCS(i+1,j), LCS(i,j+1))
    return LCS(0, 0)
```

What will happen if this code is submitted to LeetCode?

a). It will get Time Limit Exceeded as its runtime becomes exponential No

b). It will get Wrong Answer as the optimal substructure is wrong No

c). It will get Run Time Error as the base cases are off by one No

d). It will get Memory Limit Exceeded as the space complexity is too big No

e). It will get Accepted Yeah, this is an alternative solution, rated 70-30 MCQ

18. Which `LCS(A, B)` has the longest length?

a). `LCS('steven', 'seven')` 5, 'seven'; VisuAlgo default example, also easy to check

b). `LCS('awerczvxew', imijuiiynu')` 0 (nothing in common; left versus right side of QWERTY keyboard

c). `LCS('stevenhalim', 'milahnevets')` 3; this is the optional exercise 3 or 4 of lec06, finding the longest palindrome subsequence: 'eve'

d). `LCS('gatagaca', 'gattaca')` 6, 'gataca', just a grinding work, after noticing some patterns in the test cases; rated 70-30 MCQ

e). `LCS('abcdefghij', 'jihgfedcba')` 1, any single character, too many crossings

19. You have coins of denominations 1, 5, 10, and 20.

- Let $M(n)$ be the minimum number of coins to make up value n.
- Let $M^*(n)$ be the minimum number of coins to make up value n, under the constraint that you must use at least one coin of denomination 20.

Which of the following statements is **TRUE** for all $n > 20$? Correction: All of a,b,c,d are correct. Intuitively, because all of $1, 5, 10$ divide 20, so 20 should always be the preferred choice whenever $n \geq 20$, making M^* and M the same.

a). $M^*(n) = 1 + \min\{M(n-1), M(n-5), M(n-10), M(n-20)\}$

b). $M^*(n) = 1 + \min\{M^*(n-1), M^*(n-5), M^*(n-10), M(n-20)\}$ Answer, tricky options, rated 30-70 MCQ, I think students will stray to other very similar looking options...

c). $M^*(n) = 1 + \min\{M(n-1), M(n-5), M(n-10), M^*(n-20)\}$

d). $M^*(n) = 1 + \min\{M^*(n-1), M^*(n-5), M^*(n-10), M^*(n-20)\}$

e). None of the above

20. Suppose you wish to count the number of binary strings of length n that do not contain a consecutive pair of 1's (e.g., "1010" is counted, but not "0110"). For i in the range $\{0, 1, \ldots, n\}$ and b in $\{0, 1\}$, denote by $A[i][b]$ the number of strings of length i that satisfy this condition and whose last character is b. Which of the following is **TRUE**?

(a) $A[i][0] = A[i-1][1]$, and, $A[i][1] = A[i-1][0] + A[i-1][1]$

(b) $A[i][0] = A[i-1][0] + A[i-1][1] + 1$, and, $A[i][1] = A[i-2][0] + A[i-2][1]$

(c) $A[i][0] = A[i-1][0] + A[i-1][1]$, and, $A[i][1] = A[i-1][0]$ Answer; this is Fibonacci in disguise but quite tricky to see for students, rated 40-60 MCQ

(d) $A[i][0] = A[i-1][1]$, and, $A[i][1] = A[i-1][0] + A[i-2][0]$

(e) None of the above

B Essays (30 marks)

B.1 Discrete Fourier Transform (10 marks)

(Note that in this problem, it is possible to solve parts (B.1.3) and (B.1.4) WITHOUT solving (B.1.1) and (B.1.2).)

Consider a natural number n that is a power of 2, and an array of numbers A of size n. The Discrete Fourier Transform (DFT) of A is another array $\hat{A}$, also of size n, whose entries are defined are follows for $k \in \{0, \ldots, n-1\}$:

$$\hat{A}[k] = \sum_{j=0}^{n-1} A[j] \cdot (e^{-\frac{2\pi i}{n}})^{k \cdot j}$$

Above, i is the imaginary square root of 1, and $e^{-\frac{2\pi i}{n}}$ is an n-th root of unity.
What these mean are not important for this problem, except for the following important properties:

$$(e^{-\frac{2\pi i}{n}})^{\frac{n}{2}} = e^{-\pi i} = -1 \qquad (e^{-\frac{2\pi i}{n}})^2 = e^{-\frac{2\pi i}{n/2}}$$

Note that $e^{-\frac{2\pi i}{n}}$ is used in the DFT definition above because n is the size of the array A.
If its size had been some other number m, we would have used $e^{-\frac{2\pi i}{m}}$ instead.

Assume for simplicity that any arithmetic operation such as addition, multiplication, *and especially* raising e to any complex power, etc., can be performed in $O(1)$ time. Then, the most straightforward algorithm that, given A, computes its DFT $\hat{A}$ one index at a time using the definition above takes times $\Theta(n^2)$.

There is, however, a faster Divide and Conquer algorithm to compute the DFT. Consider two arrays E and O, of size $n/2$ each, that consist of the entries at the even and odd locations in A, respectively. That is, for each $j \in \{0, \ldots, n/2-1\}$:

$$E[j] = A[2 \cdot j] \qquad O[j] = A[2 \cdot j + 1]$$

We then express $\hat{A}$ in terms of $\hat{E}$ and $\hat{O}$ (the DFT's of E and O).
Note that while $\hat{A}$ has size n, the DFT's $\hat{E}$ and $\hat{O}$ have size only $n/2$.

B.1.1 Prove Part 1 (2 marks)

For any $k \in \{0, \ldots, n/2-1\}$, we have the following relationship between $\hat{A}[k]$, $\hat{E}[k]$, and $\hat{O}[k]$:

$$\hat{A}[k] = \hat{E}[k] + (e^{-\frac{2\pi i}{n}})^k \cdot \hat{O}[k]$$

Complete the following proof of the above statement,
providing adequate reasoning for each step in your derivation:

$$\begin{aligned}
\hat{A}[k] &= \sum_{j=0}^{n-1} A[j] \cdot (e^{-\frac{2\pi i}{n}})^{k \cdot j} \\
&= \sum_{j=0}^{n/2-1} A[2 \cdot j] \cdot (e^{-\frac{2\pi i}{n}})^{k \cdot 2 \cdot j} + \sum_{j=0}^{n/2-1} A[2 \cdot j + 1] \cdot (e^{-\frac{2\pi i}{n}})^{k \cdot (2 \cdot j + 1)} \\
&= \ldots \textit{(fill in the rest)} \ldots
\end{aligned}$$

$$\begin{aligned}
\hat{A}[k] &= \sum_{j=0}^{n-1} A[j] \cdot (e^{-\frac{2\pi i}{n}})^{k \cdot j} \\
&= \sum_{j=0}^{n/2-1} A[2 \cdot j] \cdot (e^{-\frac{2\pi i}{n}})^{k \cdot 2 \cdot j} + \sum_{j=0}^{n/2-1} A[2 \cdot j + 1] \cdot (e^{-\frac{2\pi i}{n}})^{k \cdot (2 \cdot j + 1)} \\
&= \sum_{j=0}^{n/2-1} E[j] \cdot (e^{-\frac{2\pi i}{n/2}})^{k \cdot j} + \sum_{j=0}^{n/2-1} O[j] \cdot (e^{-\frac{2\pi i}{n/2}})^{k \cdot j} \cdot (e^{-\frac{2\pi i}{n}})^{k} \\
&= \hat{E}[k] + (e^{-\frac{2\pi i}{n}})^{k} \cdot \hat{O}[k]
\end{aligned}$$

Grading remark (Allen Bi). Question B1.1 Most students who answered the question (excluding those leaving it blank) correctly used the property $e^{(-2\pi i/n)^2} = e^{-2\pi i/(n/2)}$ to prove the result. As long as this property is presented to derive $\hat{E}[k]$ and $\hat{O}[k]$, I considered it correct and gave full marks. One cannot derive $\hat{E}[k]$ and $\hat{O}[k]$ without using this property, as the sum is over j from 0 to $n/2 - 1$; therefore, answers not using this property were considered incorrect.

B.1.2 Prove Part 2 (2 marks)

For any $k \in \{0, \ldots, n/2 - 1\}$, we have the following relationship between $\hat{A}[k + n/2]$, $\hat{E}[k]$, and $\hat{O}[k]$:

$$\hat{A}[k + n/2] = \hat{E}[k] - (e^{-\frac{2\pi i}{n}})^{k} \cdot \hat{O}[k]$$

Complete the following proof of the above statement,
providing adequate reasoning for each step in your derivation:

$$\begin{aligned}
\hat{A}[k+n/2] &= \sum_{j=0}^{n-1} A[j] \cdot (e^{-\frac{2\pi i}{n}})^{(k+n/2)\cdot j} \\
&= \sum_{j=0}^{n/2-1} A[2j] \cdot (e^{-\frac{2\pi i}{n}})^{(k+n/2)\cdot 2j} + \sum_{j=0}^{n/2-1} A[2j+1] \cdot (e^{-\frac{2\pi i}{n}})^{(k+n/2)\cdot(2j+1)} \\
&= \sum_{j=0}^{n/2-1} A[2j] \cdot (e^{-\frac{2\pi i}{n}})^{k\cdot 2j} \cdot (e^{-\frac{2\pi i}{n}})^{\frac{n}{2}\cdot 2j} + \sum_{j=0}^{n/2-1} A[2j+1] \cdot (e^{-\frac{2\pi i}{n}})^{k\cdot 2j} \cdot (e^{-\frac{2\pi i}{n}})^{k} \cdot (e^{-\frac{2\pi i}{n}})^{\frac{n}{2}\cdot(2j+1)} \\
&= \ldots \textit{(fill in the rest)} \ldots
\end{aligned}$$

$$\begin{aligned}
\hat{A}[k+n/2] &= \sum_{j=0}^{n-1} A[j] \cdot (e^{-\frac{2\pi i}{n}})^{(k+n/2)\cdot j} \\
&= \sum_{j=0}^{n/2-1} A[2\cdot j] \cdot (e^{-\frac{2\pi i}{n}})^{(k+n/2)\cdot 2\cdot j} + \sum_{j=0}^{n/2-1} A[2\cdot j+1] \cdot (e^{-\frac{2\pi i}{n}})^{(k+n/2)\cdot(2\cdot j+1)} \\
&= \sum_{j=0}^{n/2-1} E[j] \cdot (e^{-\frac{2\pi i}{n/2}})^{k\cdot j} \cdot (e^{-\frac{2\pi i}{n}})^{\frac{n}{2}\cdot 2\cdot j} + \sum_{j=0}^{n/2-1} O[j] \cdot (e^{-\frac{2\pi i}{n/2}})^{k\cdot j} \cdot (e^{-\frac{2\pi i}{n}})^{k} \cdot (e^{-\frac{2\pi i}{n}})^{\frac{n}{2}\cdot(2\cdot j+1)} \\
&= \sum_{j=0}^{n/2-1} E[j] \cdot (e^{-\frac{2\pi i}{n/2}})^{k\cdot j} \cdot (-1)^{2j} + \sum_{j=0}^{n/2-1} O[j] \cdot (e^{-\frac{2\pi i}{n/2}})^{k\cdot j} \cdot (e^{-\frac{2\pi i}{n}})^{k} \cdot (-1)^{2j+1} \\
&= \hat{E}[k] - (e^{-\frac{2\pi i}{n}})^{k} \cdot \hat{O}[k]
\end{aligned}$$

Grading remark (Allen Bi). Question B1.2 Similarly, this problem requires using the property $e^{(-2\pi i/n)^{(n/2)}} = -1$. Students are expected to demonstrate the use of this property, along with $(-1)^{2j} = 1$ and $(-1)^{2j+1} = -1$. Applying these correctly leads to full marks. I also gave full marks to students who incorrectly used the property in B1.1 but applied the property in B1.2 correctly. The reason is that since the understanding of the property was already tested in B1.1, there should be no repeated mark deduction for the same concept. Therefore, some students get wrong with B1.1 but full marks for B1.2.

B.1.3 Divide and Conquer Algorithm (4 marks)

Construct a Divide and Conquer algorithm that uses the above observations to compute the DFT of a given array A of size n (that is a power of 2), and runs faster than $\Theta(n^2)$ (4 marks)
PS: You need to write the pseudocode for the algorithm.
You may use terms like $(e^{-\frac{2\pi i}{n}})^{a\cdot b}$ in your code *and assume* that these are computable in $O(1)$ time.
DFT(A):

1. If size$(A) = 1$: return A (do not forget the base case)
2. Construct arrays E and O of size $n/2$, where $E[j] = A[2 \cdot j]$ and $O[j] = A[2 \cdot j + 1]$
3. Compute $\hat{E} \leftarrow \text{DFT}(E)$ and $\hat{O} \leftarrow \text{DFT}(O)$ (the Divide and Conquer)
4. For $k \in \{0, \ldots, n/2 - 1\}$: set $\hat{A}[k] \leftarrow \hat{E}[k] + (e^{-\frac{2\pi i}{n}})^k \cdot \hat{O}[k]$ (basically, use Proof Part 1)
5. For $k \in \{0, \ldots, n/2 - 1\}$: set $\hat{A}[k + n/2] \leftarrow \hat{E}[k] - (e^{-\frac{2\pi i}{n}})^k \cdot \hat{O}[k]$ (use Proof Part 2)
6. return $\hat{A}$

Grading remark (Prof Prashant). Some relatively minor mistakes, like not specifying the base case, were ignored. Pseudo-code was needed in the answer, and answers that did not include pseudo-code or an equally unambiguous description of the algorithm were not accepted. A common error was splitting the array into its first and second halves, rather than according to odd and even indices.

A few answers that made significant mistakes but contained the most important ideas were given partial marks, but these were exceptions. Most common among these were answers that specified correctly the algorithm for computing $\hat{A}$ given $\hat{E}$ and $\hat{O}$, but failed to make the correct recursive calls to compute $\hat{E}$ and $\hat{O}$.

B.1.4 Running Time (2 marks)

Compute the running time of the algorithm you presented in subsection B.1.3 above!
Given an instance of size n, the algorithm recursively calls itself on two instances of size $n/2$, and then performs $O(n)$ work before returning. The running time $T(n)$ follows the recursion:

$$T(n) = 2 \cdot T(n/2) + O(n)$$

Using the Master Theorem or simply by comparing to other such algorithms like merge sort, this implies that the running time is $T(n) = O(n \log n)$.

Grading remark (Prof Prashant). It is required that you provide adequate arguments to support your conclusion of the running time, such as the recursive relation above or a tree-based argument. If the answer to the previous part is blank or contains an algorithm that is so poorly specified that it is not possible to unambiguously determine its complexity, it is not possible to score any marks in this part.

Students who had given an incorrect algorithm for the previous part but evaluated the complexity of their algorithm correctly were given marks. Note that this is not necessarily our grading policy in other problems, and often we do not give marks for the complexity analysis if the algorithm itself is wrong.

B.2 Random Greedy Maximal Independent Set (10 marks)

Let $G = (V, E)$ be a *simple* undirected graph (no self-loops and no parallel edges). Consider the following randomized procedure. Choose a uniformly random permutation π of V. Process vertices in the order π, maintaining a set $S \subseteq V$ initialized to $\varnothing$ (empty set). When a vertex v is processed, add v to S if none of its already-processed neighbors was previously added to S. At the end, output S.

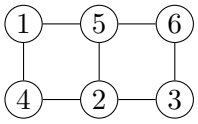


Figure 1: To see how the procedure works, let us look at a simple example of 2×3 grid graph. Each vertex is labeled by its rank in a randomly chosen permutation. In this run, the algorithm produces the set $S = \{1, 2, 6\}$. Of course, the outcome depends on the permutation: For instance, if the vertices are processed in the order $6 \to 5 \to 4 \to 3 \to 2 \to 1$, then the procedure instead outputs $S = \{4, 6\}$.

B.2.1 Complete Graphs (2 marks)

A *complete graph* K_n is an n-vertex simple graph in which each pair of vertices is connected by an edge. What is the size of S when $G = K_n$? You only need to provide the final answer; no proof or explanation is required.

Answer: $|S| = 1$.

Grading remark (Prof Chang Yi-Jun). No other answer is accepted (e.g., writing $1 \leq |S| \leq n$ is zero mark). Some very careful students wrote $|S| = \begin{cases} 1, & n \geq 1 \\ 0, & n = 0 \end{cases}$, which is of course correct.

B.2.2 Complete Bipartite Graphs (4 marks)

A *complete bipartite graph* $K_{a,b}$ is a simple graph whose vertex set is partitioned into two parts L and R with $|L| = a$ and $|R| = b$ such that there are no edges within L or within R, and every vertex of L is adjacent to every vertex of R. Prove that for $G = K_{a,b}$,

$$\mathbb{E}\big[\,|S|\,\big] \;=\; \frac{a^2 + b^2}{a + b}.$$

Let x be the vertex with smallest rank under π. If $x \in L$, then x is added to S. Every vertex $r \in R$ has x as a neighbor, so when r is processed, it finds a neighbor already in S and is therefore *never* added. Conversely, any $l \in L$ has neighbors only in R, and since no vertex from R is ever added, each $l \in L$ is added when processed. Hence, conditioning on $x \in L$, the output is $S = L$ and $|S| = a$. If $x \in R$, the symmetric argument shows $S = R$ and $|S| = b$. Since $\Pr[x \in L] = a/(a+b)$ and $\Pr[x \in R] = b/(a+b)$, we obtain

$$\mathbb{E}[\,|S|\,] \;=\; \frac{a}{a+b} \cdot a \;+\; \frac{b}{a+b} \cdot b \;=\; \frac{a^2 + b^2}{a+b}.$$

Grading remark (Prof Chang Yi-Jun). To get full mark, you need to correctly explain the calculation. For example, if you just write $\mathbb{E}[\,|S|\,] = \frac{a}{a+b} \cdot a + \frac{b}{a+b} \cdot b = \frac{a^2+b^2}{a+b}$ without any explanation, then we are not sure if you truly understand the calculation.

B.2.3 Lower Bound (4 marks)

Prove that the lower bound

$$\mathbb{E}[\,|S|\,] \geq \sum_{v \in V} \frac{1}{\deg(v)+1}$$

holds for any graph, where $\deg(v)$ denotes the degree of v (i.e., its number of neighbors).

For each $v \in V$, let A_v be the event that v appears *before all of its neighbors* in the random order π. If A_v occurs, then when v is processed none of its neighbors has yet been processed (hence none is in S), so v is added to S. Therefore,

$$\mathbf{1}[\,v \in S\,] \geq \mathbf{1}[\,A_v\,] = \Pr[A_v] \qquad \text{for every } v \in V.$$

Summing over v and taking expectations,

$$\mathbb{E}[\,|S|\,] = \mathbb{E}\left[\sum_v \mathbf{1}[v \in S]\right] \geq \sum_v \Pr[A_v].$$

Among the $\deg(v)+1$ vertices in $\{v\} \cup N(v)$, all relative orders are equally likely, so $\Pr[A_v] = 1/(\deg(v)+1)$. Hence

$$\mathbb{E}[\,|S|\,] \geq \sum_{v \in V} \frac{1}{\deg(v)+1}.$$

Grading remark (Prof Chang Yi-Jun). Here are some common mistakes observed:

- Considering only complete graphs and/or complete bipartite graphs. (You need to consider all graphs.)
- Stating that exactly one vertex among v and its neighbors joins S, and claiming that this implies that $1/(\deg(v)+1)$ is the probability that v joins S. (This statement is incorrect.)
- Failing to recognize that the linearity of expectation can be applied, leading to unnecessarily complicated and incorrect inclusion–exclusion calculations.
- Claiming an equality instead of a lower bound. (Full marks were still awarded for this.)

Figure 2: Examples of complete graphs and complete bipartite graphs.

B.3 Longest Contiguous Increasing Stock Prices (10 marks)

You are given three floating-point (`double` data type) arrays `price1`, `price2`, and `price3`, each of length n, representing the stock prices of your favorite blue-chip company based in Singapore (D05) over the last n days, as reported by three different market makers.

You want to analyze a trading strategy using this historical data. Your strategy depends on finding the length of the **longest contiguous strictly increasing stock prices**, assuming you can switch between market makers on any day at no cost.

More formally, define a "combined" ~~integer~~ array `price` of length n. For each index $i \in [1..n]$, you may assign `price[i]` to be *either* `price1[i]`, `price2[i]`, or `price3[i]`. Your goal is to find the length of the **longest contiguous non-empty subarray** of `price` such that the prices are strictly increasing.

An example is shown in Table 1.

	1	2	3	4	5	6	7	8	9
`price1`	50.75	**50.48**	50.47	50.52	**50.85**	**50.88**	50.80	51.39	51.53
`price2`	50.76	50.46	**50.54**	50.53	50.80	50.83	50.85	51.38	51.54
`price3`	50.77	50.49	50.81	**50.81**	50.79	50.81	**51.40**	51.37	51.55

Table 1: Example of the last $n = 9$ days.
All floating-point values start with 5 and a '.' in the middle, so focus on the last three digits.
The 6 days of contiguous strictly increasing D05 stock prices are **highlighted** (other solutions exist).

B.3.1 Subtask 1: One `price1` only; Design an $O(n \log n)$ algorithm (3 marks)

Suppose there is only **one** market maker that you trust, providing `price1` only. For example, see Table 2. Design an $O(n \log n)$ algorithm to solve Subtask 1.

	1	2	3	4	5	6	7	8	9
`price1`	50.75	50.48	**50.47**	**50.52**	**50.85**	**50.88**	50.80	51.39	51.53

Table 2: Example of the last $n = 9$ days (only `price1`).
The 4 days of contiguous strictly increasing D05 stock prices are **highlighted** (unique solution).

Simple $\Theta(n)$ is possible and the $O(n \log n)$ is to mask this simple target time complexity, assume answer is 1 (minimal answer anyway) and set $current\_length = 1$, then just loop from 1 to $n - 1$. For

each ADJACENT (that is, contiguous) elements, see if $price1[i-1] < price1[i]$, if it is, we extend the length by 1, otherwise we reset to 1. We keep the largest running *current_length* that we see throughout the process and report this as the answer.

Grading remark (Alvin Yan). Here are some common mistakes observed:

- Most answers correctly gave $O(n)$ time algorithm to solve the problem, both answers describing the algorithm and pseudocode were accepted. A small number correctly gave divide and conquer algorithms using $O(n \log n)$ time, which was to divide the array into half, and return the maximum of the three cases: longest contiguous increasing subarray strictly in left, longest contiguous increasing subarray strictly in right, and longest contiguous strictly increasing subarray that must start in the left half and end in the right half -¿ explicitly stating to do this by simply doing a linear scan from the two endpoints of the middle. Answers in pseudocode/code/prose were all accepted as long as it was clear enough.
- Common mistakes: Some answers did not correctly track the maximum length contiguous increasing subarray, and overwrote previously stored longer length found.
- Some answers did not have a line that sets the count of the maximum length contiguous increasing subarray to 1 when it was detected that $prices1[i] \leq prices1[i-1]$.
- These answers were given some benefit of doubt that it is a minor mistake and accepted if the algorithm idea was correct, and no other issues.
- A frequent minor issue that was ignored is that answers would be off by 1 as they reset the count of the maximum length contiguous increasing subarray to 0 instead of 1 when $prices1[i] \leq prices1[i-1]$.
- A significant number of answers attempted a divide and conquer approach given the $O(n \log n)$ time given in the question statement. Most were incorrect, for example suggesting in the merge step to combine the two subarrays obtained from the recursive step only if they were adjacent indices. However, this does not guarantee that the longest contiguous increasing subarray that starts in the left and ends in the right will be found, which might be the longest. Many other answers were too vague or handwavy in explaining how to do the merge step and what to return, which were not accepted.

B.3.2 Subtask 2: Design an $o(n \log n)$ algorithm (5 marks)

Design an algorithm to solve the general version of task B.3 (the one with *3 (THREE)* different prices for each given day). To be given marks, your algorithm must runs in little $o(n \log n)$ – supply a proof of correctness. You can use additional space (as much as you need, as long as it is reasonable).

Let's abbreviate the **Longest-Contiguous-Increasing-Stock-Prices** problem as `LCISP` and we have put `price1`, `price2`, and `price3` values into an $3 \times n$ array `prices` (to simplify the discussion below).

The expected solution is to use Dynamic Programming (DP). The state is (i, j) where i is the current day $i \in [0..n-1]$ (the original problem uses $[1..n]$ but we switch to 0-based version) and $j \in [0, 1, 2]$ (again, we use 0-based version) is one of the market maker (this second parameter is **very important**). We define $LCISP(i, j)$ to be the **length of the** `LCISP` from the first day 0 until day i, ending at `prices[j][i]` of market maker j (at day i) – without the second parameter, we do NOT know this information.

$LCISP(n-1, j) = 1$ as on the last day, we cannot extend any further and the longest that we can have is 1.

$LCISP(i, j)$ is the maximum of either 1 (cannot extend) or $1 + LCISP(i+1, k)$ if `prices[j][i]` $<$ `prices[k][i+1]`, $\forall k \in [0, 1, 2]$ (i.e., we stay at the same market maker, or instantly switch to one of the other two). On the 'can extend' case, one can quickly argue using cut-and-paste argument that $LCISP(i+1, k)$ must also be an optimal subproblem too, otherwise we can use the 'more optimal' subproblem and plus 1, to have a more optimal $LCISP(i, j)$ - contradiction, so $LCISP(i+1, k)$ (when we can extend) must be optimal too.

The time complexity of this approach, if computed using top-down DP (or bottom-up DP) – exponential otherwise as there are many overlapping subproblems – is $n \times 3$ different subproblems times 3, or $9 \times n \in Theta(n)$, which fits the $o(n \log n)$ requirement.

The standard DP (top-down or bottom-up) requires $\Theta(n \cdot 3) \in \Theta(n)$ additional space, but we can still optimize this further (see the next subsection).

Grading remark (Prof Halim). Here are some common mistakes observed:

- Many students tried to extend their B.3.1 answer into B.3.2 + B.3.3, usually by using some form of greedy, it will not work. Try countering your own 'greedy' (when not on time pressure).
- The tell-tale sign is missing parameter $j \in [0, 1, 2]$, thus at a given day i, the algorithm cannot differentiate which market maker that the stock is currently on.

B.3.3 Subtask 3: Solve Subtask 2 with only $\Theta(1)$ additional space (2 marks)

Now this is the hardest version of this task.
Please ensure that you are confident with your Subtask B.3.2 solution first.
For the last 2 marks, your correct algorithm must run in little $o(n \log n)$ and only use $\Theta(1)$ additional space (note that the $\Theta(n)$ input size to store the prices is *not* considered in the analysis).

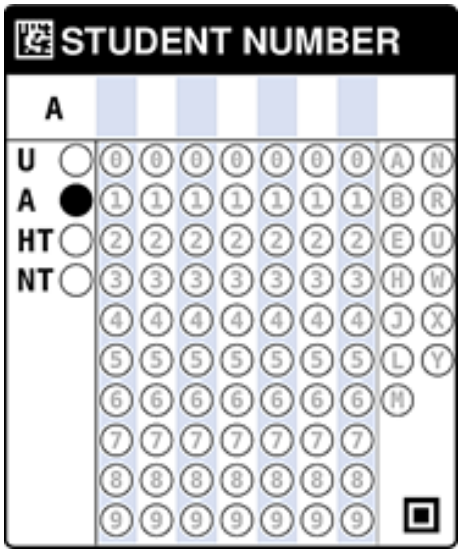
The optimal substructure computation is only between the previous day $i-1$ and the next day i. It is possible to just store the results of the last two days (2×3 variables) instead of $n \times 3$ variables. This is called bottom-up-DP with computation on-the-fly technique. It is designed to differentiate a few students who understand this space-saving technique.

Grading remark (Prof Halim). Here are some common mistakes observed:

- Students who gave wrong answer in part B.3.2 will not get anything in B.3.3 (as there are many wrong algorithms that only uses constant additional space.

The Answer Sheet

Write your Student Number in the box below using **(2B) pencil** and shade the corresponding bubbles. **Do NOT write your name**.



Fill in your MCQ answers in the special MCQ answer box below for automatic grading.
We do not manually check your answer.
Shade your answer properly (use (2B) pencil, fully enclose the circle; select just one circle).

No	1	2	3	4	5	6	7	8	9	10
A	◯	◯	◯	◯	◯	◯	◯	◯	◯	◯
B	◯	◯	◯	◯	◯	◯	◯	◯	◯	◯
C	◯	◯	◯	◯	◯	◯	◯	◯	◯	◯
D	◯	◯	◯	◯	◯	◯	◯	◯	◯	◯
E	◯	◯	◯	◯	◯	◯	◯	◯	◯	◯

No	11	12	13	14	15	16	17	18	19	20
A	◯	◯	◯	◯	◯	◯	◯	◯	◯	◯
B	◯	◯	◯	◯	◯	◯	◯	◯	◯	◯
C	◯	◯	◯	◯	◯	◯	◯	◯	◯	◯
D	◯	◯	◯	◯	◯	◯	◯	◯	◯	◯
E	◯	◯	◯	◯	◯	◯	◯	◯	◯	◯

Box B.1.1. Prove Part 1

Box B.1.2. Prove Part 2

Box B.1.3. Divide and Conquer Algorithm

Box B.1.4. The Running Time of D&C Algorithm in B.1.3

Box B.2.1. Random Greedy Maximal Independent Set - Complete Graphs

Box B.2.2. Random Greedy Maximal Independent Set - Complete Bipartite Graphs

Box B.2.3. Random Greedy Maximal Independent Set - Lower Bound

Box B.3.1. Longest Contiguous Increasing Stock Prices - One price1 only, $O(n \log n)$ algorithm

Box B.3.2. Longest Contiguous Increasing Stock Prices - Design an $o(n \log n)$ algorithm

Box B.3.3. Longest Contiguous Increasing Stock Prices - only $\Theta(1)$ additional space

– END OF PAPER; All the Best –