Average-Case Fine-Grained Hardness

Marshall Ball   Alon Rosen   Manuel Sabin   Prashant Nalini Vasudevan

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- APSP
- Orthogonal Vectors

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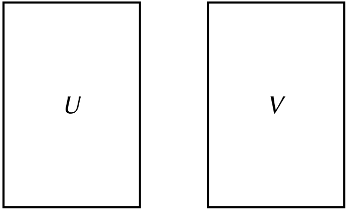
- Necessary for cryptography
- Potential use in algorithm design

Plan

- Introduce problems
- Present average-case reduction
- Summarise
- Present Proof of Work
- ???
- Profit.

Worst-Case: Orthogonal Vectors



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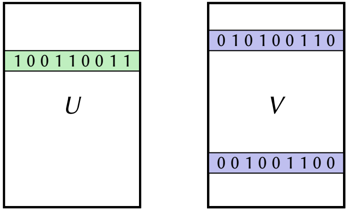

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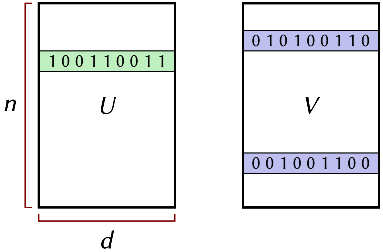

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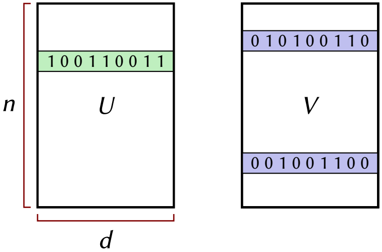

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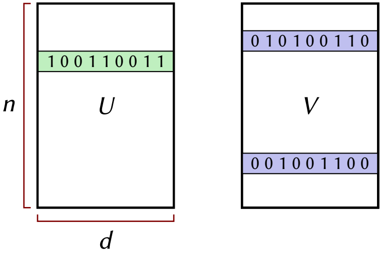

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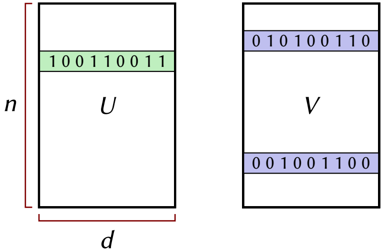

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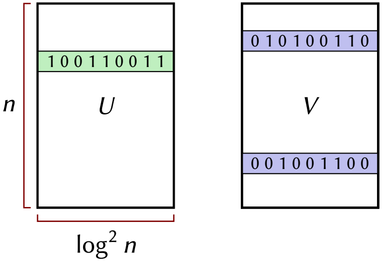

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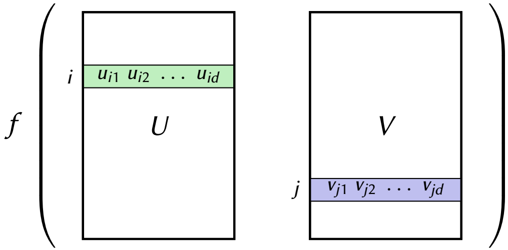

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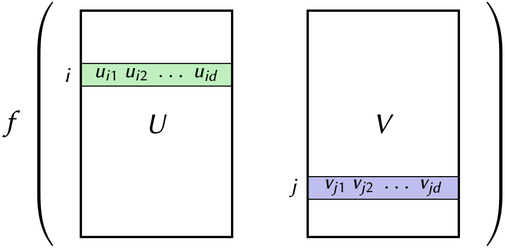

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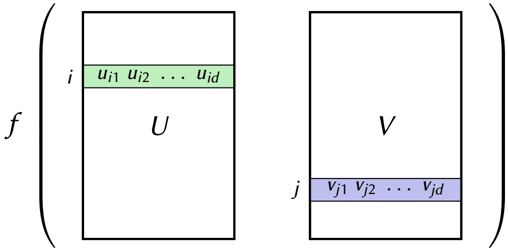

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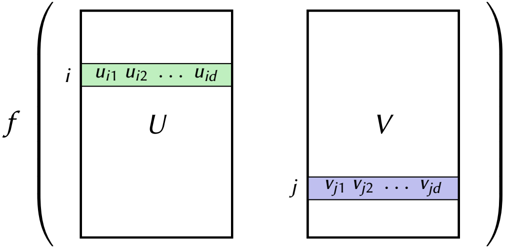

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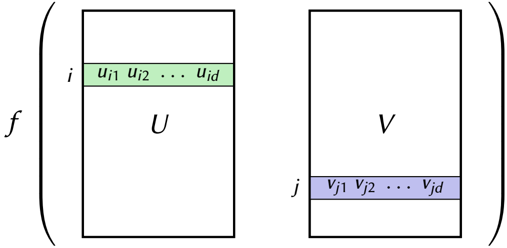

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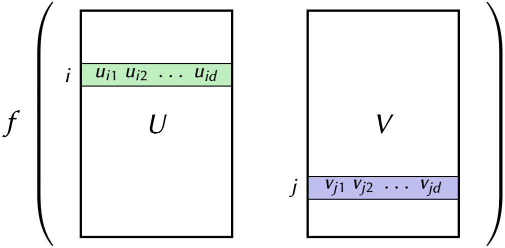

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$$\boxed{\begin{array}{c} deg(f) = 2d \\ d = \log^2 n \end{array}}$$

Worst-Case to Average-Case

Theorem

$$\exists A \text{ in time } n^{1+\alpha} : \Pr_{x \leftarrow \mathbb{F}_p^{2nd}}[A(x) = f(x)] \geq \frac{1}{n^{o(1)}}$$

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Corollary

OV takes $n^{2-o(1)} \Rightarrow f$ takes $n^{2-o(1)}$ on *average*

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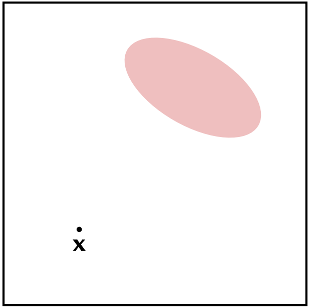

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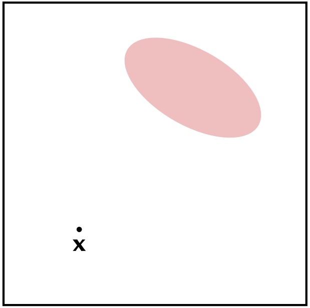

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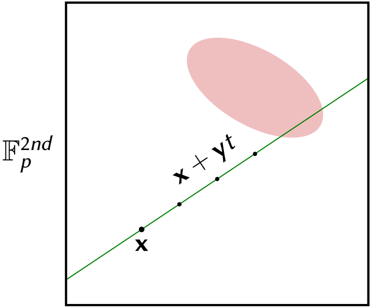

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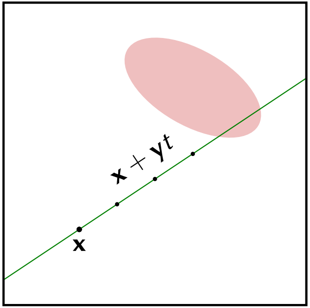

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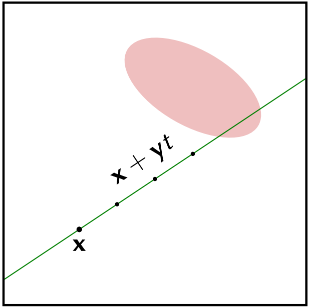

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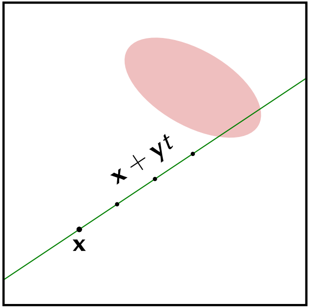

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Theorem [Wil16]

There is an MA proof system for proving $(f(\mathbf{x}) = y)$ that has:

- perfect completeness and negligible soundness.
- prover complexity $\widetilde{O}(n^2)$.
- verifier complexity $\widetilde{O}(n)$.

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(See [DN92] for generic constructions and applications.)

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  - Design algorithms to evaluate polynomials that work on average.
- Better reductions
  - Is it actually possible to do better than guessing at random?

To be passed in case of an abundance of time.

k-SAT and SETH

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Strong Exponential Time Hypothesis (SETH) [IPZ98]

$\forall \epsilon \; \exists k$: k-SAT takes $\widetilde{\Omega}(2^{(1-\epsilon)n})$ time.

An Efficient MA Protocol for f [Wil16]

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- Verification:
  - Check r at random point. (Computation of ϕ and correct value – $\widetilde{O}(n)$)
  - Compute $r(i)$ for $i \in [n]$ and sum to get $f(U, V)$. (Batch evaluation – $\widetilde{O}(n)$)

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