The shortest path problem

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Input: A Directed graph
$G = (V, E)$

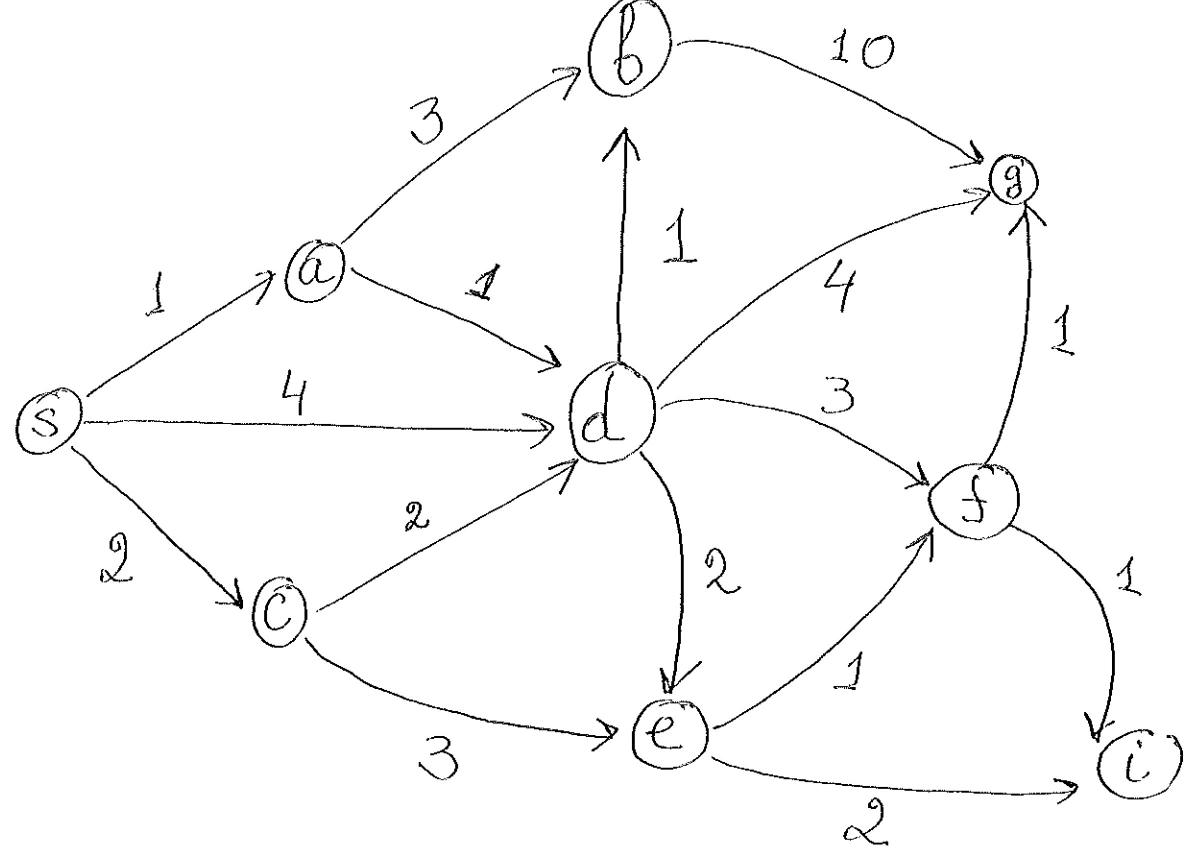
Start vertex: s

The cost l_e of each edge e.

For a path P,
$l(P)$ = the sum of all edge costs of P.

Goal: Find the shortest ~~paths~~ path from s to all other vertices of V.

Example 1:



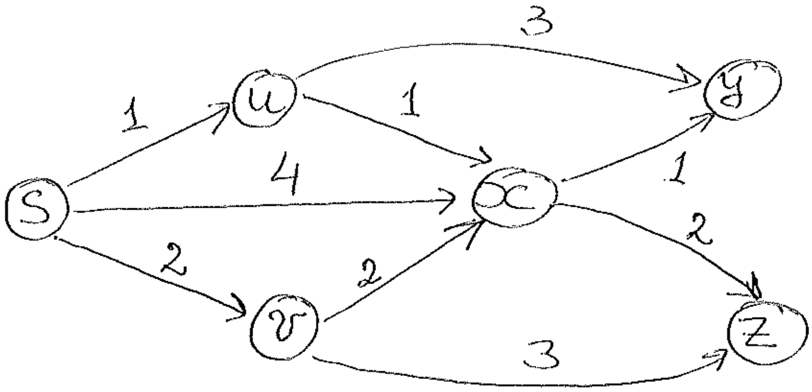
P_1: s, d, f, i;

$l(P) = 8$

P_2: s, a, d, f, i; $l(P_2) = 6$

P_3: s, c, e, i; $l(P_3) = 7$

Example 2:



$S = \{s\}, \quad d(s) = 0$

$S = \{s, u\}, \quad d(u) = 1$

$S = \{s, u, v\}, \quad d(v) = 2$

$S = \{s, u, v, x\}, \quad d(x) = 2$

$S = \{s, u, v, x, y\}, \quad d(y) = 3$

$S = \{s, u, v, x, y, z\}, \quad d(z) = 4$

Dijkstra Algorithm (G, s):

Initially $S = \{s\}$, $d(s) = 0$.

While $S \neq V$

Select a $v \notin S$ with at least one edge from set S for which

$$d'(v) = \min_{\substack{u \in S \\ (u,v) \in E}} d(u) + \ell(u,v)$$

is as small as possible.

Add v to S.

Define $d(v) = d'(v)$.

To analyze the algorithm we define the path P_u for each $u \in G$.

We use the algorithm.

For $s \in S$, at the initial stage, P_s is just s.

Let v be the node added to S at some stage of the algorithm. Let (u,v) be the edge for which

$$\min_{\substack{u \in S \\ (u,v) \in E}} d(u) + \ell(u,v)$$

is achived. Then P_v is the path P_u followed by v.

Property. For each v, the path P_v is a shortest path from s to v.

The proof is by induction on the number k of iterations of the while loop of the algorithm.

When $k=0$, we have

P_s : s.

Clearly, P_s is the shortest to s.

Suppose, by the end of iteration k, we have proved the property for all $u \in S$.

Consider v added to S at stage $k+1$. Consider the path P_v. It is formed by adding edge (u,v) to P_u.

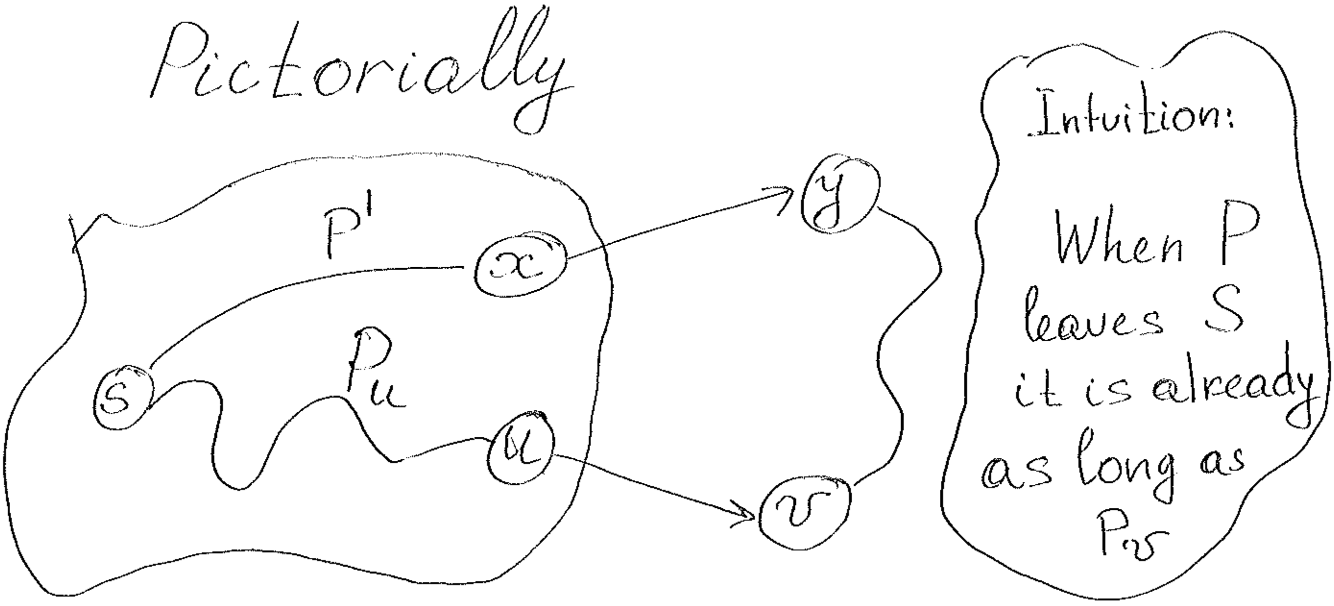
Want to show that P_v is shortest.

Consider any path P from s to v.

Goal: $\ell(P_v) \leq \ell(P)$.

The path P, in order to reach v, must leave S at some point. Let (x,y) be the first edge on P such that $x \in S$ and $y \notin S$.

Pictorially



So we have:

$$\ell(P) \geq$$

$$\ell(P') + \ell(x,y) \geq d(x) + \ell(x,y) \geq$$

$$\geq d'(y) \geq d'(v) = d(v) = \ell(P).$$

The second inequality uses inductive hypothesis, the third the definition of d', the fourth the definition of v.