Approximation Algorithms

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Introduction

Geng Xue

Optimization problem

Find the minimum/maximum of …

– Vertex Cover : A minimum set of vertices that covers all the edges in a graph.

NP optimization problem

- The hardness of NP optimization is NP hard.

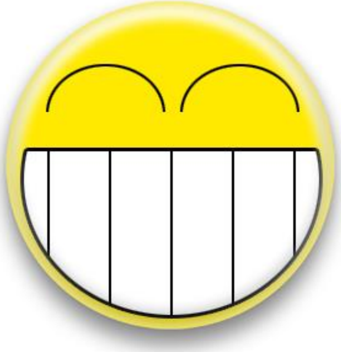
The hardest problem in NP

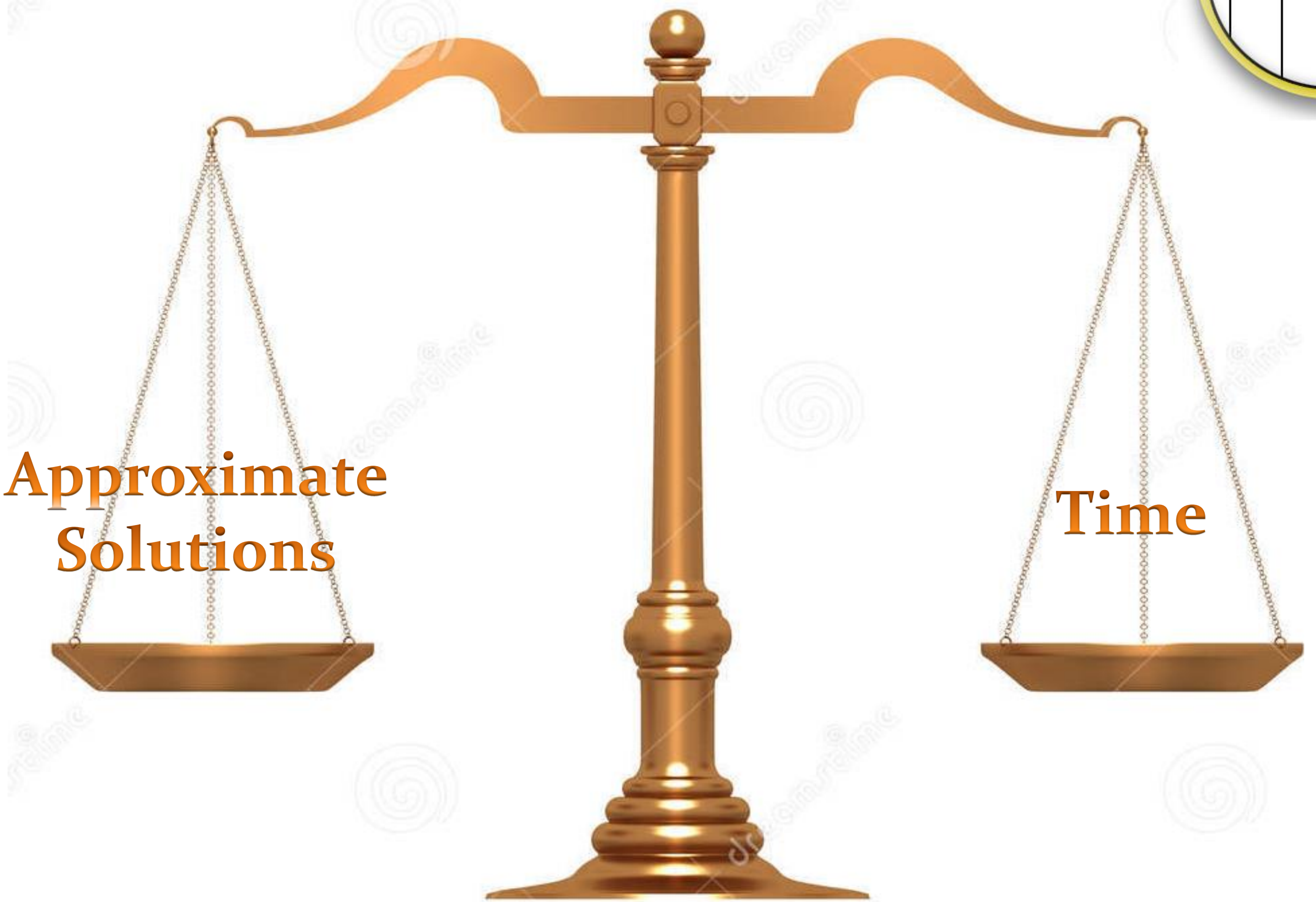
- Can't be solved in polynomial time.

Time consuming !!!

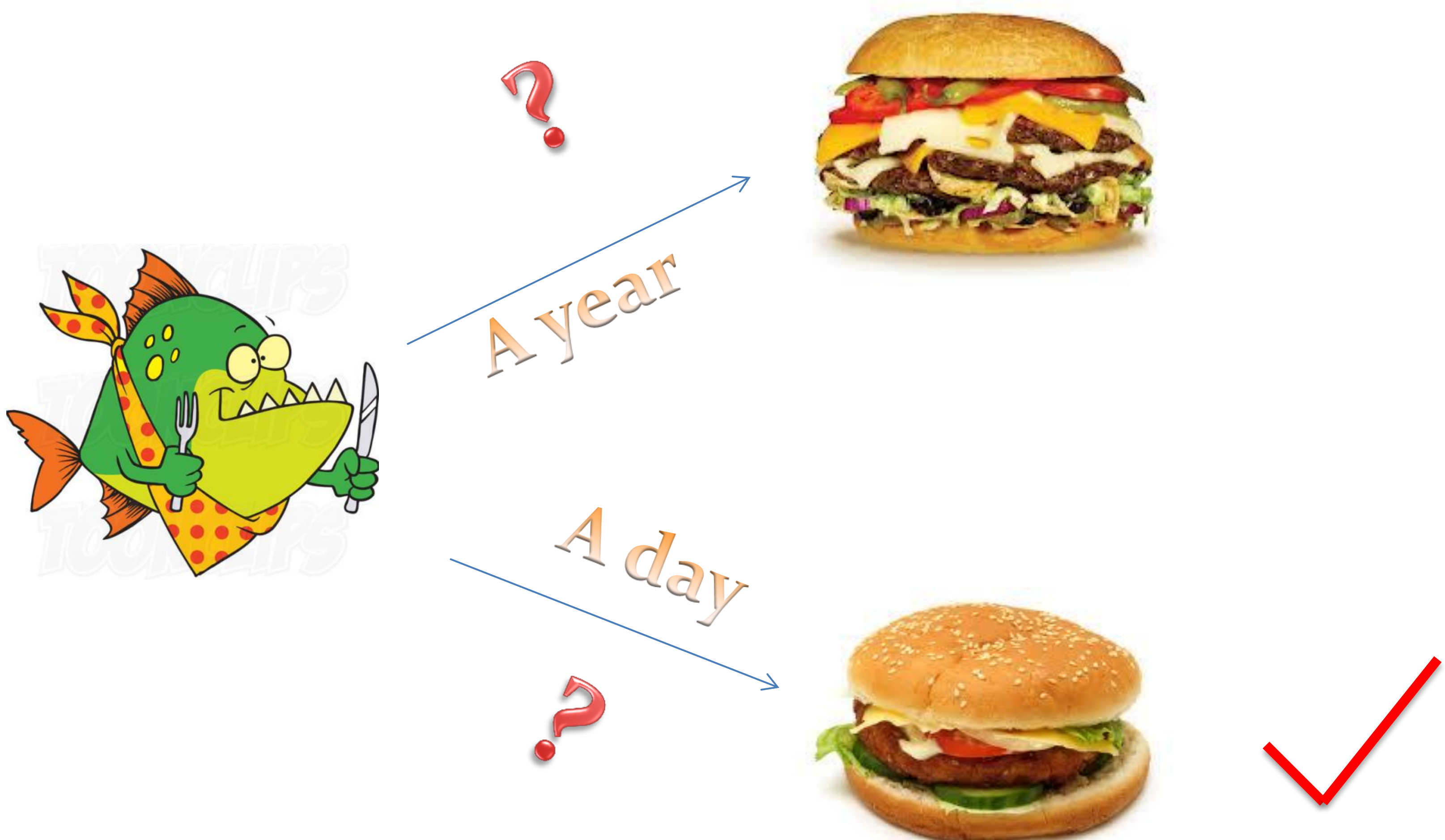
How to solve ?

How to solve ?





How to solve ?



How to solve ?

Approximation Algorithm!

- **sub-optimal solution!**
- **Polynomial Time!**

How to evaluate ?

Mine is better!

Standard!

How to evaluate ?

OPT

Exact optimal value of optimization problem.

Factor

relative performance guarantee.

- Sub-optimal solution : $\rho \times OPT$ (ρ is factor).
- The closer factor ρ is to 1, The better the algorithm is.

Set cover

- It leads to the development of fundamental techniques for the entire approximation algorithms field.
- Due to set cover, many of the basic algorithm design techniques can be explained with great ease.

What is set cover ?

- Definition:
  - Given a universe U of n elements, a collection of subsets of U, $S = \{S_1, S_2, \dots, S_k\}$, and a cost function $c: S \rightarrow Q^+$, find a minimum cost sub-collection of S that covers all elements of U.

What is set cover ?

- Example:

S	Sets	Cost
S_1	{1,2}	1
S_2	{3,4}	1
S_3	{1,3}	2
S_4	{5,6}	1
S_5	{1,5}	3
S_6	{4,6}	1

Universal Set :
$U = \{1,2,3,4,5,6\}$

Find a sub-collection of S that covers all the elements of U with minimum cost.

Solution?

$S_1 \quad S_2 \quad S_4$

Cost = 1 + 1 + 1 = 3

Approximation algorithms to set cover problem

- Combinatorial algorithms
- Linear programming based Algorithms (LP-based)

Combinatorial & LP-based

- Combinatorial algorithms
  - Greedy algorithms
  - Layering algorithms
- LP-based algorithms
  - Dual Fitting
  - Rounding
  - Primal–Dual Schema

Greedy set cover algorithm

Cai Jingli

Greedy set cover algorithm

1. $C \leftarrow \emptyset$
2. While $C \neq U$ do

   Find the most cost-effective set in the current iteration, say S.

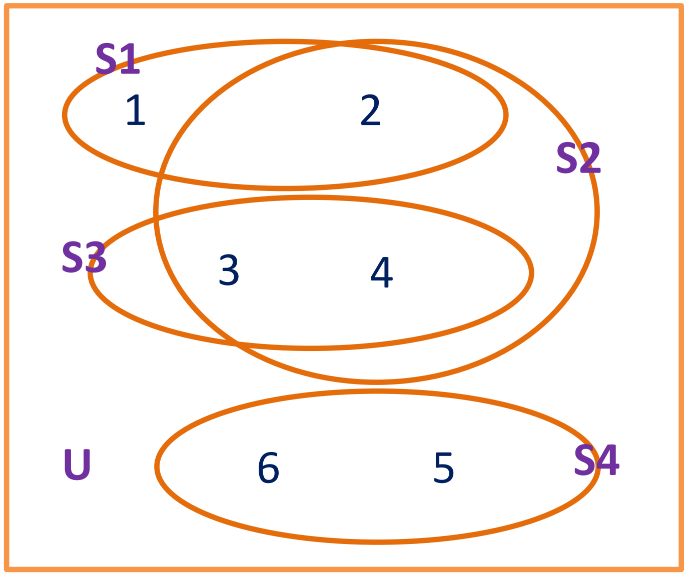
   Let $\alpha = \frac{\mathrm{cost}(S)}{|S-C|}$, i.e., the cost-effectiveness of S.

   Pick S, and for each $e \in S - C$, set $\mathrm{price}(e) = \alpha$.

   $C \leftarrow C \cup S$.
3. Output the picked sets.

Where C is the set of elements already covered at the beginning of an iteration and α is the average cost at which it covers new elements.

Example



subset	cost
S1 = {1, 2}	1
S2 = {2, 3, 4}	2
S3 = {3, 4}	3
S4 = {5, 6}	3

Iteration 1: C = {}

α1 = 1 / 2 = 0.5

α2 = 2 / 3 = 0.67

α3 = 3 / 2 = 1.5

α4 = 3 / 2 = 1.5

C = {1, 2}

Price(1) = 0.5

Price(2) = 0.5

Iteration 2: C = {1, 2}

α2 = 2 / 2 = 1

α3 = 3 / 2 = 1.5

α4 = 3 / 2 = 1.5

C = {1, 2, 3, 4}

Price(3) = 1

Price(4) = 1

Iteration 3: C = {1, 2, 3, 4 }

α3 = 3 / 0 = infinite

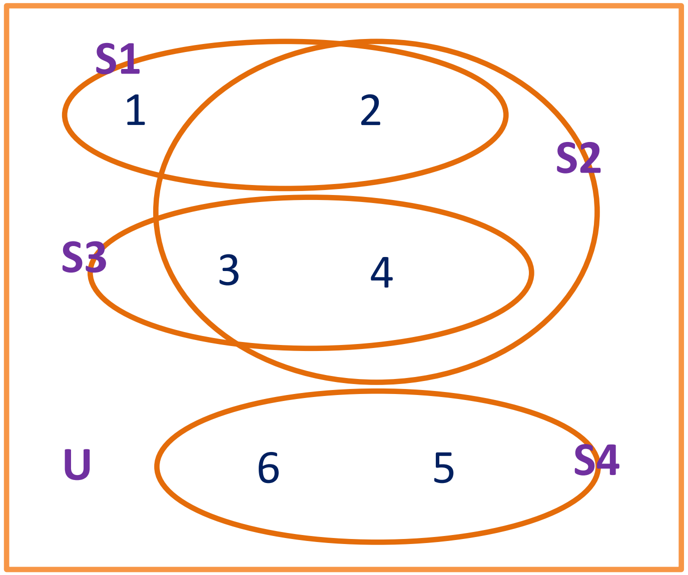
α4 = 3 / 2 = 1.5

C = {1, 2, 3, 4, 5, 6}

Price(5) = 1.5

Price(6) = 1.5

Example



subset	cost
S1 = {1, 2}	1
S2 = {2, 3, 4}	2
S3 = {3, 4}	3
S4 = {5, 6}	3

The picked sets = {S1, S2, S4}

Total cost = Cost(S1)+Cost(S2)+Cost(S4)

= Price(1) + Price(2) + … + Price(6)

= 0.5 + 0.5 + 1 + 1 + 1.5 + 1.5 = 6

Theorem

- The greedy algorithm is an $\boldsymbol{H_n}$ factor approximation algorithm for the minimum set cover problem, where

$$H_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$$

$$\sum_i Cost(Si) \leq H_n \cdot OPT$$

Proof

1) We know $\sum_{e \in U} price(e) =$ cost of the greedy algorithm $= c(S_1) + \text{c}(S_2) + \cdots + c(S_m)$.
2) We will show $price(e_k) \le \frac{OPT}{n-k+1}$, where e_k is the kth element covered.

If 2) is proved then theorem is proved also.

$$\sum_{i} Cost(Si) = \sum_{e \in U} price(e) \le \sum_{k=1}^{n} \frac{OPT}{n-k+1} = OPT * \sum_{k=1}^{n} \frac{1}{n-k+1}$$

$$= OPT * (1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}) = H_n \cdot OPT$$

$$price(e_k) \le \frac{OPT}{n-k+1}$$

- Say the optimal sets are $O_1, O_2, \dots, O_p$, so $\mathrm{OPT} = \mathrm{c}(O_1) + c(O_2) + \cdots + c\left(O_p\right) = \sum_{i=1}^{p} \mathrm{c}(O_i)$
- Now, assume the greedy algorithm has covered the elements in C so far.

$$|U - C| \le |O_1 \cap (U - C)| + \cdots + \left|O_p \cap (U - C)\right|$$

$$= \sum_{i=1}^{p} |O_i \cap (U - C)|$$

- $\mathrm{price}(e_k) = \alpha \le \frac{c(O_i)}{|O_i \cap (U-C)|}$ (2), *i*=1,…,*p*. we know this because of greedy algorithm

$$price(e_k) \leq \frac{OPT}{n-k+1}$$

$$(1)\ \mathrm{OPT} = \sum_{i=1}^{p} \mathrm{c}(O_i)$$

$$(2) |U-C| \leq \sum_{i=1}^{p} |O_i \cap (U-C)|$$

$$(3) c(O_i) \geq \alpha \cdot |O_i \cap (U-C)|$$

- So $OPT = \sum_{i=1}^{p} \mathrm{c}(O_i) \geq \alpha \cdot \sum_{i=1}^{p} |O_i \cap (U-C)| \geq \alpha \cdot |U-C|$

$$|U-C| = n-k+1$$

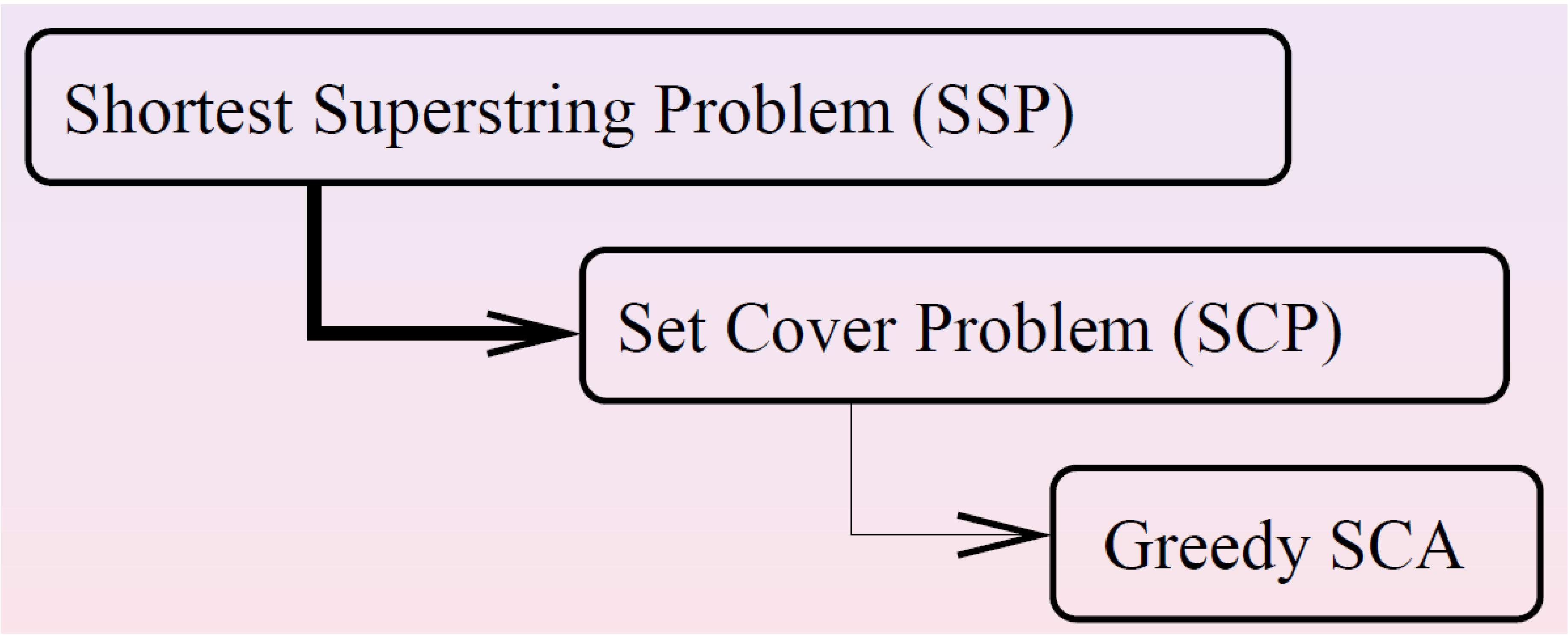
$$price(e_k) = \alpha \leq \frac{OPT}{n-k+1}$$

Shortest Superstring Problem (SSP)

- Given a finite alphabet Σ, and set of n strings, $S = \{s_1, \dots, s_n\} \subseteq \Sigma^+$.
- Find a shortest string s that contains each s_i as a substring.
- Without loss of generality, we may assume that no string s_i is a substring of another string s_j , $j \neq i$.

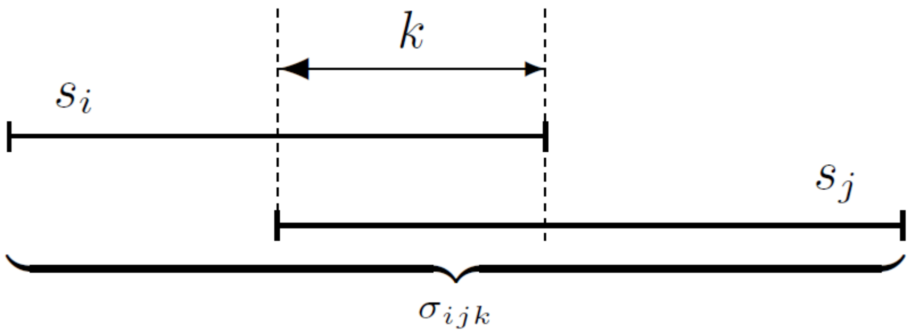
Application: Shotgun sequencing

Approximating SSP Using Set Cover



Set Cover Based Algorithm

- **Set Cover Problem:**
  - Choose some sets that cover all **elements** with least cost
- **Elements**
  - The input strings
- **Subsets**
  - σ_{ijk} = string obtained by overlapping input strings s_i and s_j , with k letters.
  - $\beta = \boldsymbol{S} \cup \sigma_{ijk},\ all\ i,j,k$
  - Let $\pi \in \beta$
  - **set**(π) = $\{s \in \boldsymbol{S} \mid s$ is a substr. of $\pi\}$



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  - Let $\pi \in \beta$
  - **set**(π) = $\{s \in \boldsymbol{S} \mid s$ is a substr. of $\pi\}$
- **Cost of a subset**
  - set(π) is $|\pi|$
- A solution to SSP is the concatenation of π obtained from SCP

Example

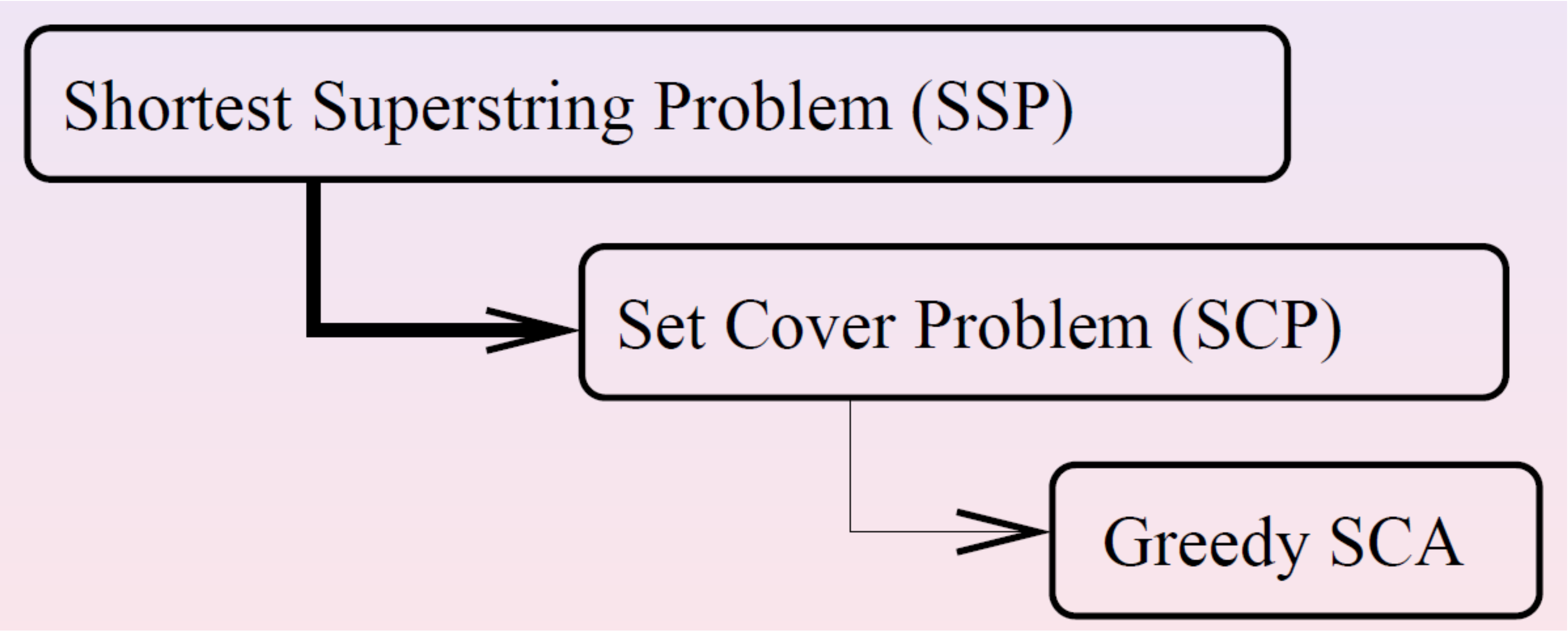
- S = {CATGC, CTAAGT, GCTA, TTCA, ATGCATC}

π	Set	Cost
CATGC . . . . CTAAGT CATGCTAAGT	CATGC, CTAAGT, GCTA	10
CATGC . . . . . GCTA CATGCTA	CATGC, GCTA	7
. . . . CATGC ATGCATC . . . . . ATGCATCATGC	CATGC, ATGCATC	11
CTAAGT . . . . TTCA CTAAGTTCA	CTAAGT, TTCA	9
ATGCATC . . . . CTAAGT ATGCATCTAAGT	CTAAGT, ATGCATC	12

GCTA . . . . ATGCATC GCTATGCATC	GCTA, ATGCATC	10
TTCA . . . . ATGCATC TTCATGCATC	TTCA, ATGCATC, CATGC	10
GCTA . . . . CTAAGT GCTAAGT	GCTA, CTAAGT	7
TTCA . . . . . CATGC TTCATGC	CATGC, TTCA	7
CATGC . . . . ATGCATC CATGCATC	CATGC, ATGCATC	8
CATGC	CATGC	5
CTAAGT	CTAAGT	6
GCTA	GCTA	4
TTCA	TTCA	4
ATGCATC	ATGCATC	7

Lemma

$$OPT \leq OPT_{SCA} \leq 2H_n \cdot OPT$$



Layering Technique

Xing Zhe

Layering Technique

Set Cover Problem

Combinatorial algorithms

Greedy: H_n

Layering ?

Vertex Cover Problem

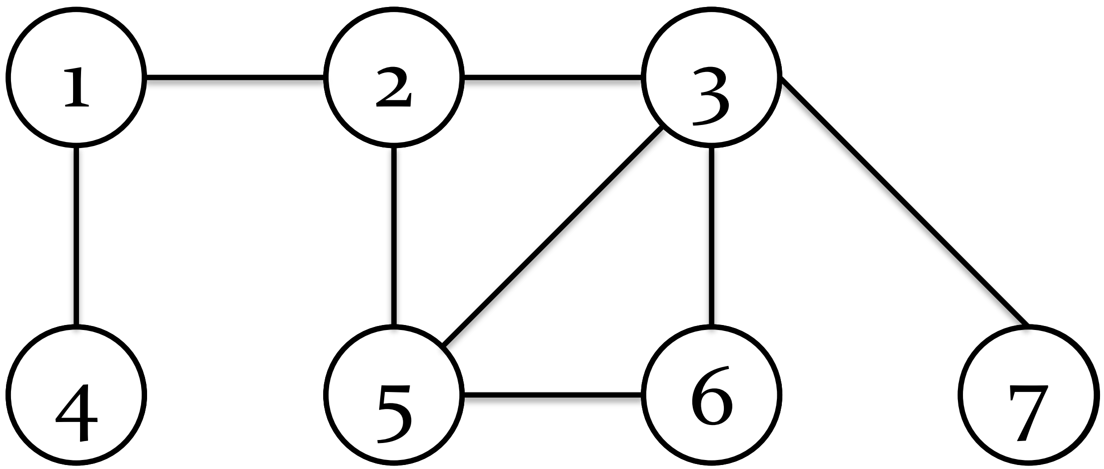
Definition:

Given a graph $G = (V, E)$, and a weight function $w\colon V \to Q^+$ assigning weights to the vertices, find a minimum weighted subset of vertices $C \subseteq V$, such that C "covers" all edges in E, i.e., every edge $e_i \in E$ is incident to at least one vertex in C.

Vertex Cover Problem

Example:

all vertices of unit weight



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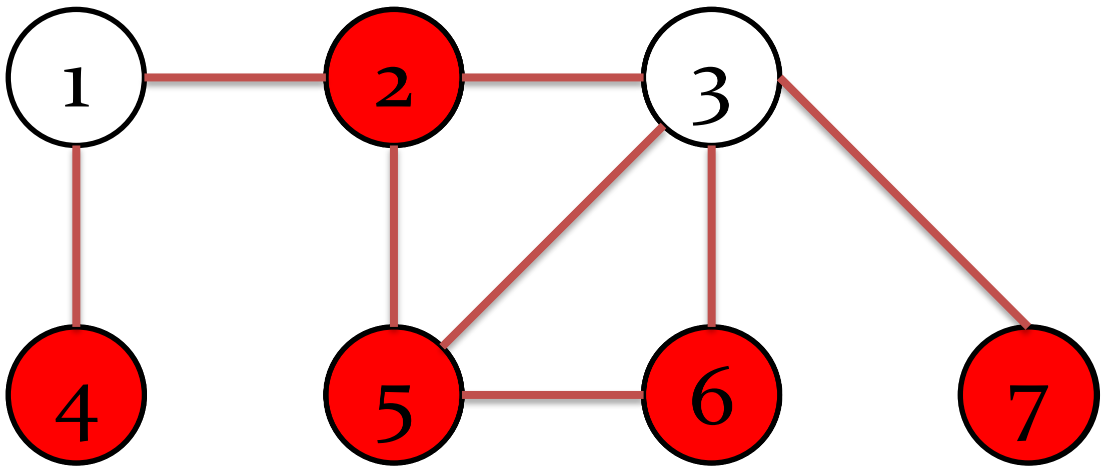
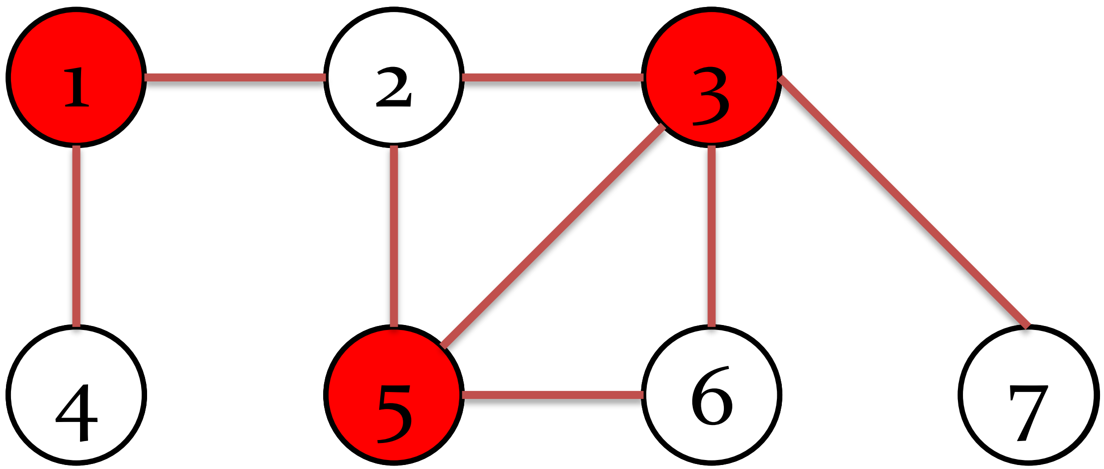


Total cost = 5

Vertex Cover Problem

Example:

all vertices of unit weight

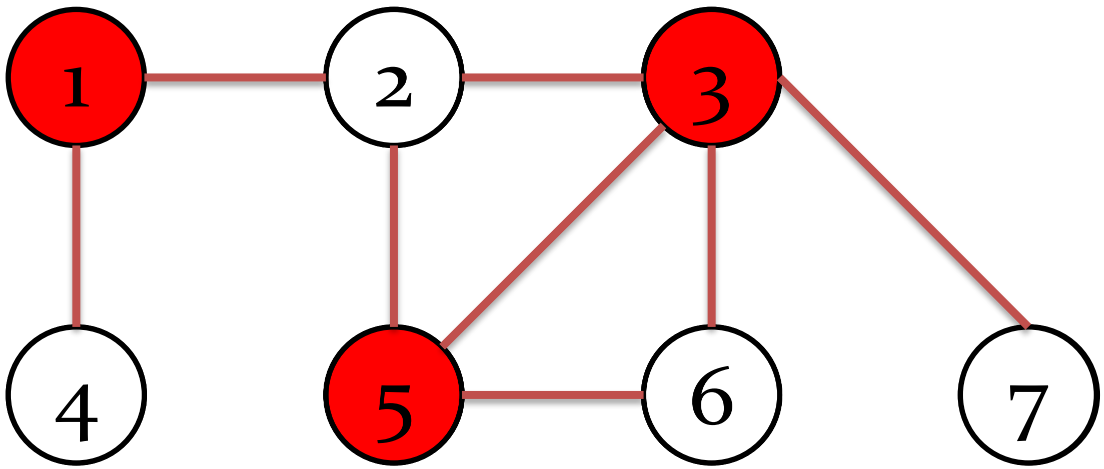


Total cost = 3

Vertex Cover Problem

Example:

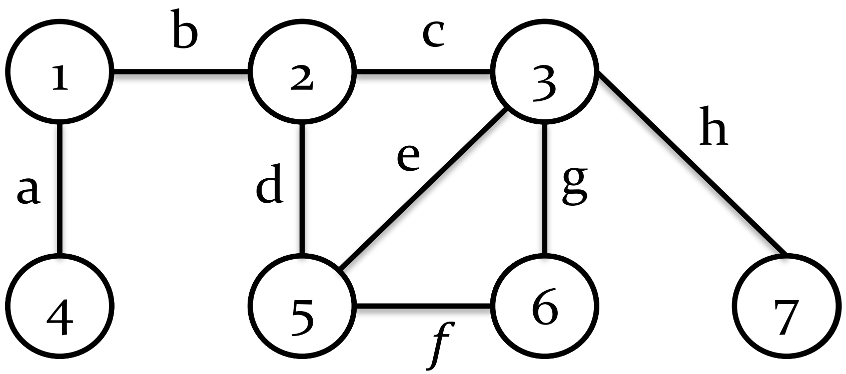
all vertices of unit weight



Total cost = 3 ⟵ **OPT**

Vertex Cover Problem

Vertex cover is a special case of set cover.



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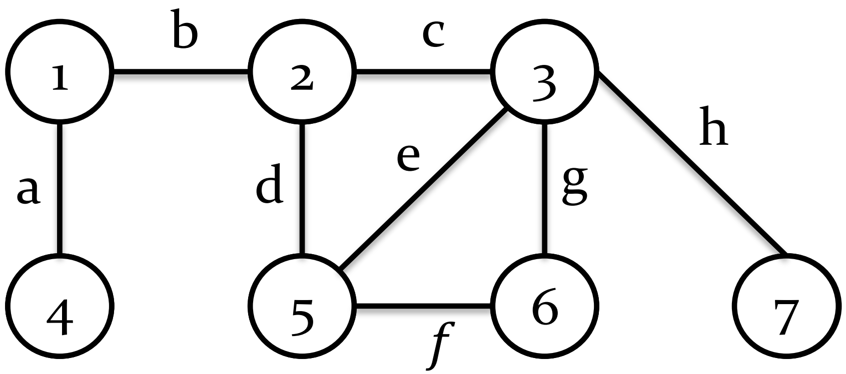

Edges are **elements**, **vertices** are **subsets**

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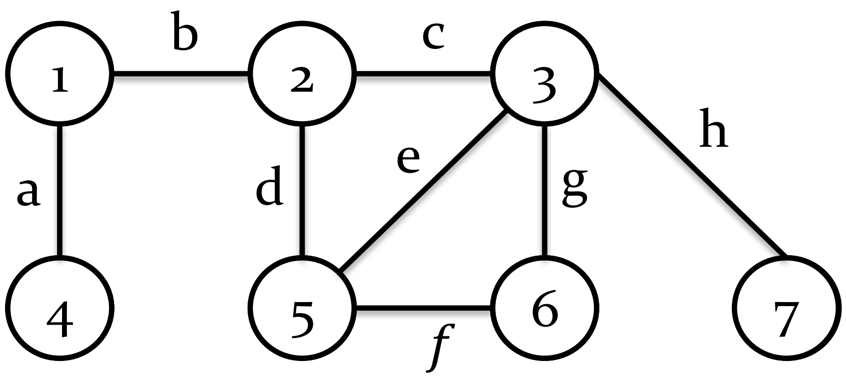

Edges are **elements**, **vertices** are **subsets**

Universe $U = \{a, b, c, d, e, f, g, h\}$

Vertex Cover Problem

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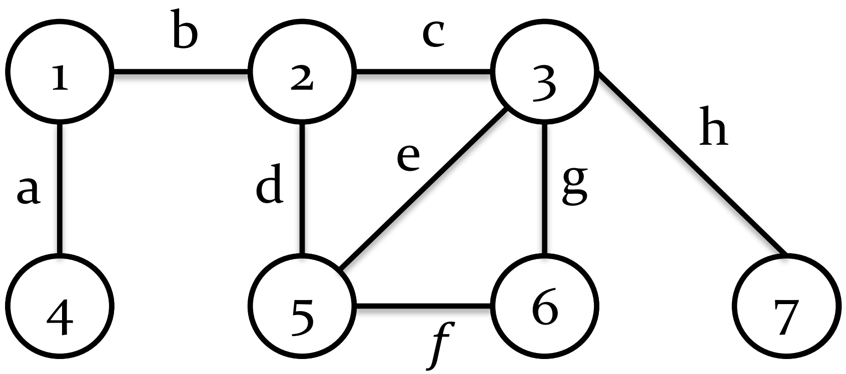

$$S_1 = \{a, b\}$$
$$S_2 = \{b, c, d\}$$
$$S_3 = \{c, e, g, h\}$$
$$S_4 = \{a\}$$
$$S_5 = \{d, e, f\}$$
$$S_6 = \{f, g\}$$
$$S_7 = \{h\}$$

Edges are **elements**, **vertices** are **subsets**

Universe $U = \{a, b, c, d, e, f, g, h\}$

A collection of subsets $S = \{S_1, S_2, S_3, S_4, S_5, S_6, S_7\}$

Notation

1) frequency

2) f : the frequency of the most frequent element.

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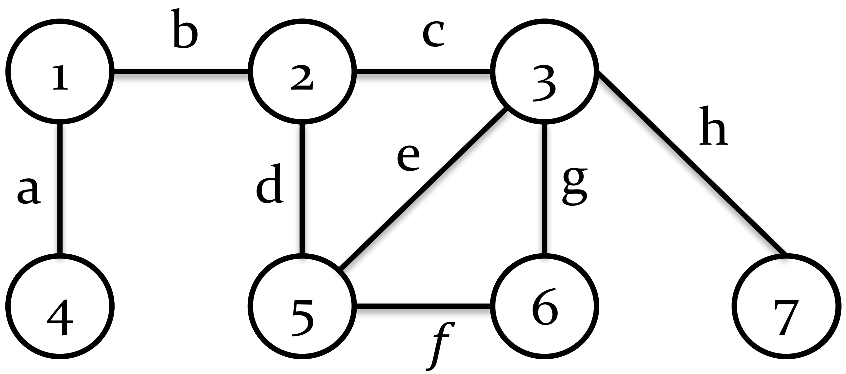
In the vertex cover problem, $f = ?$

Vertex Cover Problem

Vertex cover is a special case of set cover.



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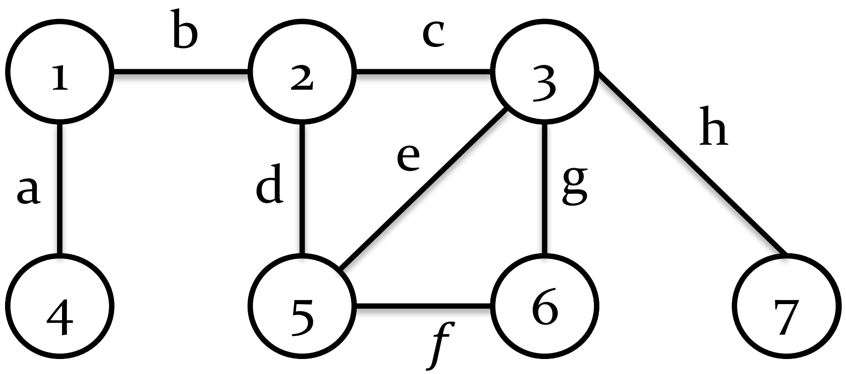

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Edges are **elements**, **vertices** are **subsets**

Universe $U = \{a, b, c, d, e, f, g, h\}$

A collection of subsets $S = \{S_1, S_2, S_3, S_4, S_5, S_6, S_7\}$

Vertex cover problem is a set cover problem with $\boldsymbol{f = 2}$

Approximation factor

1) frequency

2) f : the frequency of the most frequent element.

In the vertex cover problem, $f = 2$

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1) frequency

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In the vertex cover problem, $f = 2$

	Set Cover Problem	Vertex Cover Problem
Layering Algorithm	**factor** $= f$	**factor** $= 2$

Vertex Cover Problem

arbitrary weight function:

$$w: V \to Q^+$$

Vertex Cover Problem

arbitrary weight function:

$$w: V \to Q^+$$

degree-weighted function:

$$\exists\, c > 0 \text{ s.t.}$$

$$\forall\, v \in V, \quad w(v) = c \cdot \deg(v)$$

Degree weighted function

Lemma:

If $w\colon V \to Q^{+}$ is a degree-weighted function.

Then $w(V) \leq \mathbf{2} \cdot OPT$

Lemma:

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Lemma:

If $w\colon V \rightarrow Q^+$ is a degree-weighted function.

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Proof:

degree-weighted function, $w(v) = c \cdot \deg(v)$

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C^* is the optimal vertex cover in G, $\sum_{v \in C^*} \deg(v) \geq |E|$

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$OPT = w(C^*) = \mathrm{c} \cdot \sum_{v \in C^*} \deg(v) \geq c \cdot |E|$ **(1)**

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in worst case, we pick all vertices.

handshaking lemma, $\sum_{v \in V} \deg(v) = 2|E|$

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$w(V) = \sum_{v \in V} w(v) = \sum_{v \in V} c \cdot \deg(v) = c \cdot \sum_{v \in V} \deg(v) = c \cdot 2|E|$ **(2)**

Lemma:

If $w\colon V \to Q^{+}$ is a degree-weighted function.

Then $w(V) \leq 2 \cdot OPT$

Proof:

degree-weighted function, $w(v) = c \cdot \deg(v)$

C^* is the optimal vertex cover in G, $\sum_{v \in C^*} \deg(v) \geq |E|$

$OPT = w(C^*) = \mathrm{c} \cdot \sum_{v \in C^*} \deg(v) \geq c \cdot |E|$ **(1)**

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$w(V) = \sum_{v \in V} w(v) = \sum_{v \in V} c \cdot \deg(v) = c \cdot \sum_{v \in V} \deg(v) = c \cdot 2|E|$ **(2)**

from (1) and (2), $w(V) \leq 2 \cdot OPT$

Layering algorithm

Basic idea:

arbitrary weight function

↓

several degree-weighted functions

↓

nice property (*factor* = 2) holds in each layer

Layering algorithm

1) remove all degree zero vertices, say this set is $\boldsymbol{D_0}$

2) compute $\boldsymbol{c} = \boldsymbol{min}\{\boldsymbol{w}(\boldsymbol{v})/\boldsymbol{deg}(\boldsymbol{v})\}$

3) compute degree-weighted function $\mathbf{t}(\boldsymbol{v}) = \boldsymbol{c} \cdot \mathbf{deg}(\boldsymbol{v})$

4) compute residual weight function $\boldsymbol{w}'(\boldsymbol{v}) = \boldsymbol{w}(\boldsymbol{v}) - \boldsymbol{t}(\boldsymbol{v})$

5) let $\boldsymbol{W_0} = \{\boldsymbol{v} \mid \boldsymbol{w}'(\boldsymbol{v}) = \mathbf{0}\}$, pick zero residual vertices into the cover set

6) let $\boldsymbol{G_1}$ be the graph induced on $\boldsymbol{V} - (\boldsymbol{D_0} \cup \boldsymbol{W_0})$

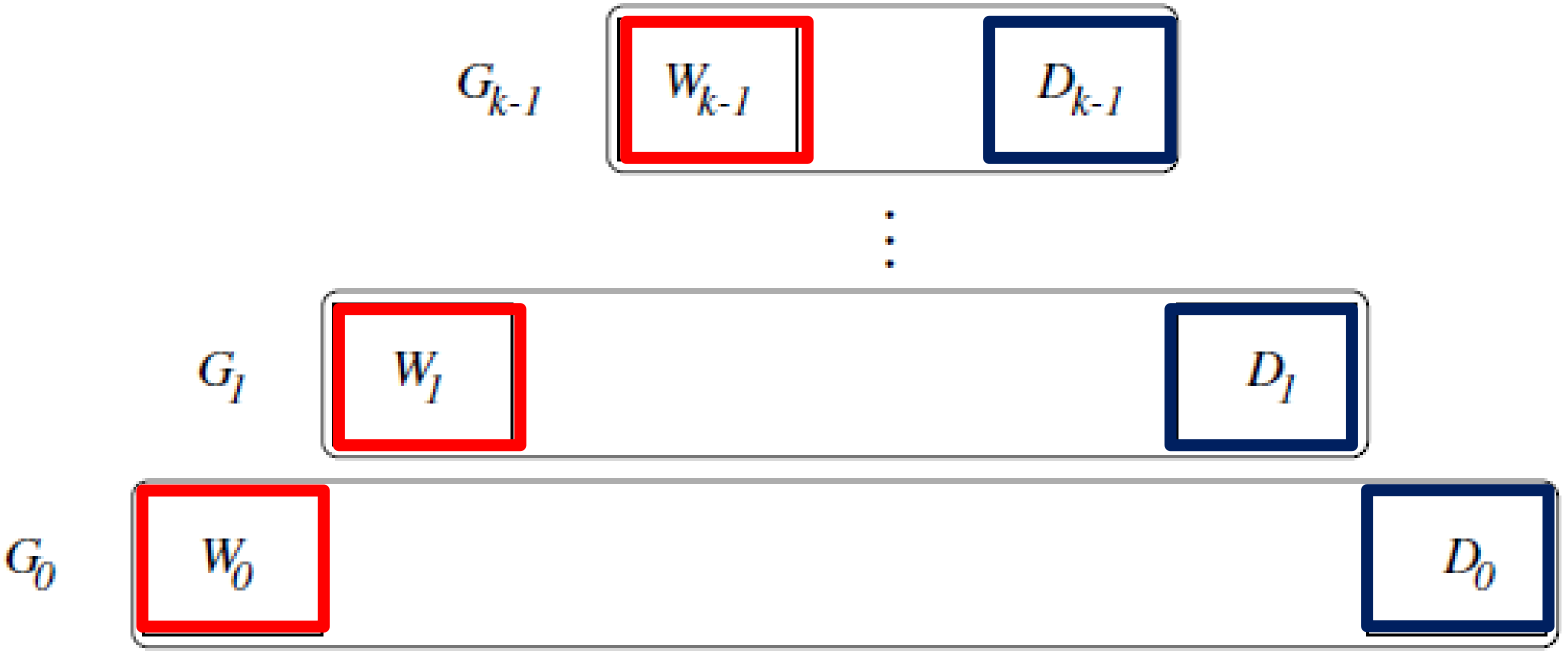
7) repeat the entire process on $\boldsymbol{G_1}$ w.r.t. the **residual** weight function,

until all vertices are of degree zero

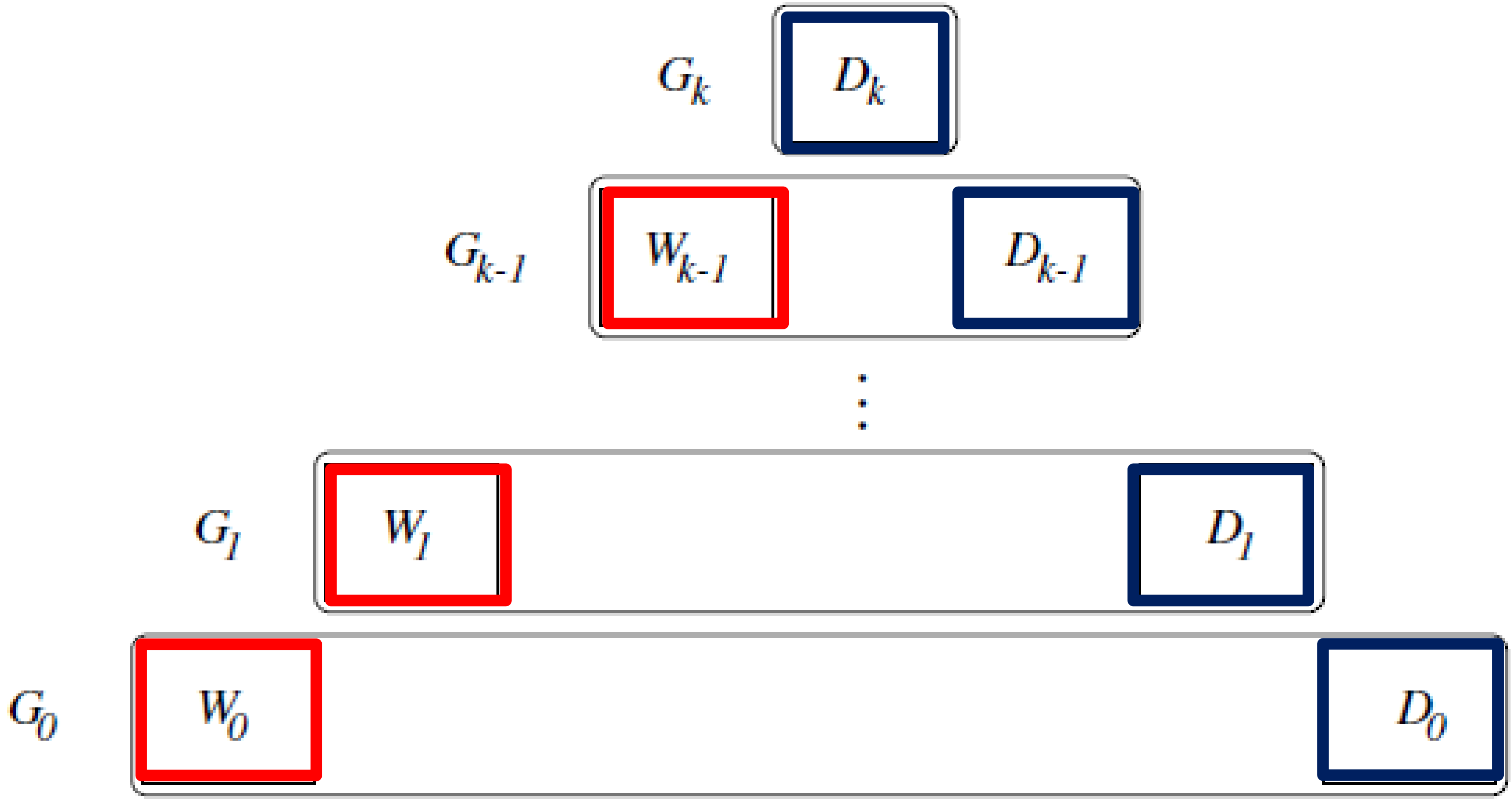
Layering algorithm

Layering algorithm

Layering algorithm



Layering algorithm



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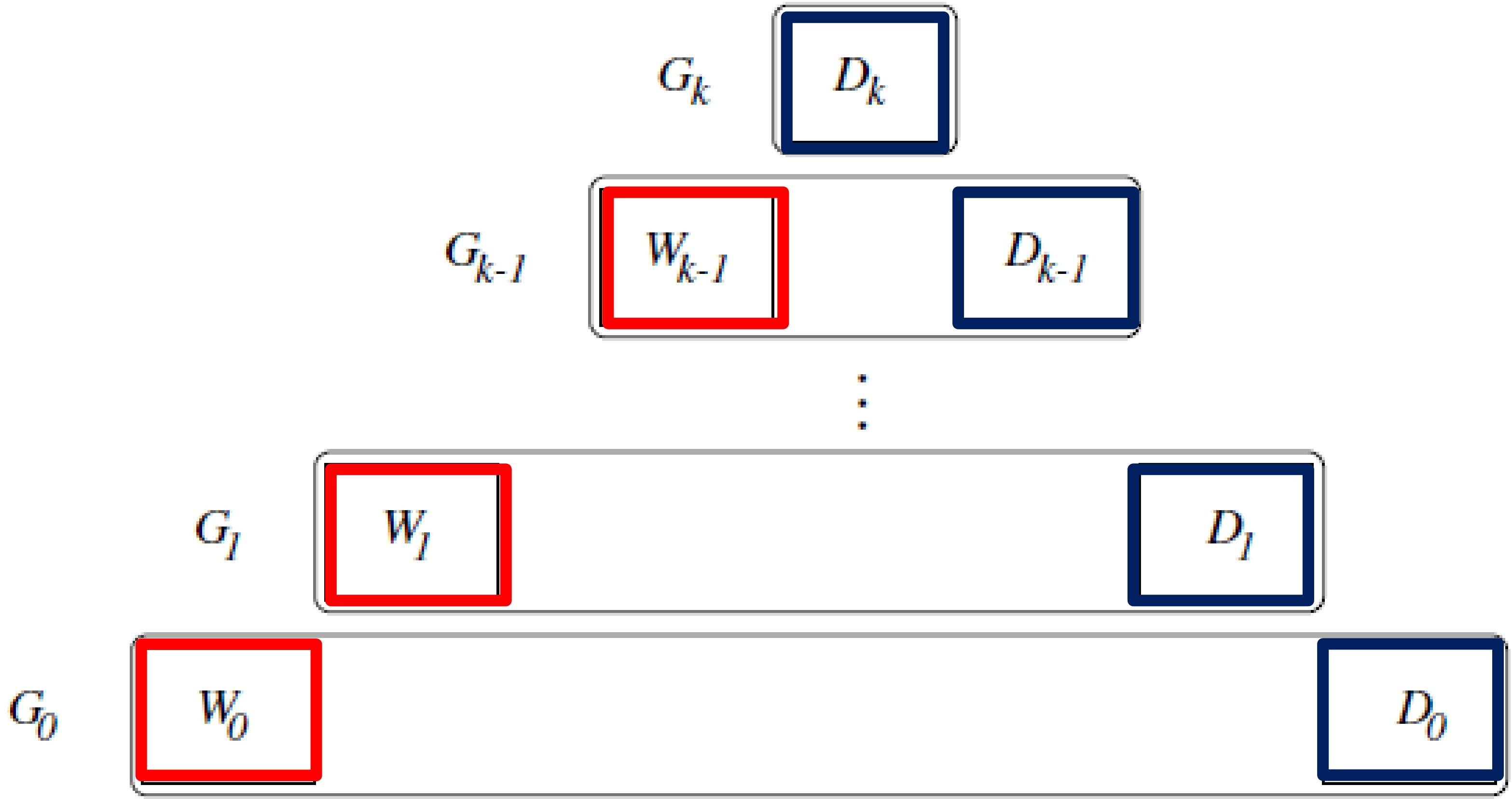


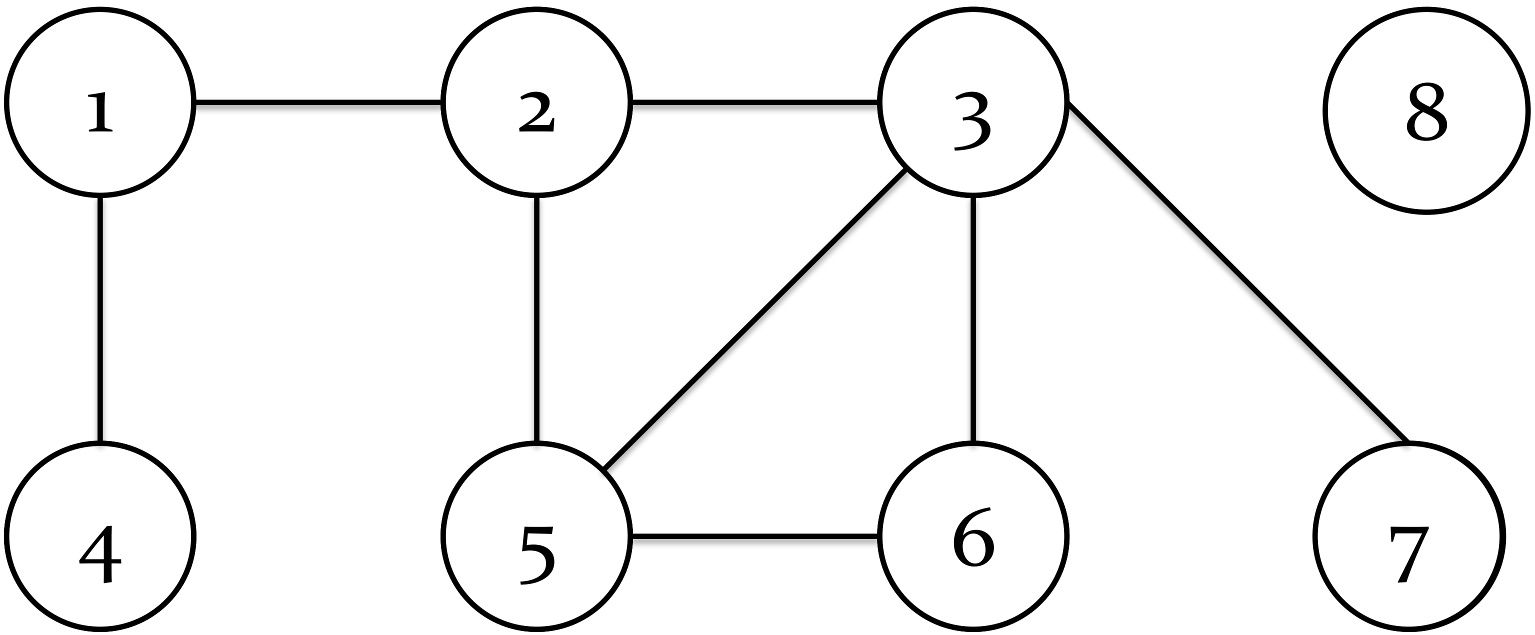
The vertex cover chosen is $\boldsymbol{C} = \boldsymbol{W_0} \cup \boldsymbol{W_1} \cup \ldots \cup \boldsymbol{W_{k-1}}$

Clearly, $\boldsymbol{V} - \boldsymbol{C} = \boldsymbol{D_0} \cup \boldsymbol{D_1} \cup \ldots \cup \boldsymbol{D_k}$

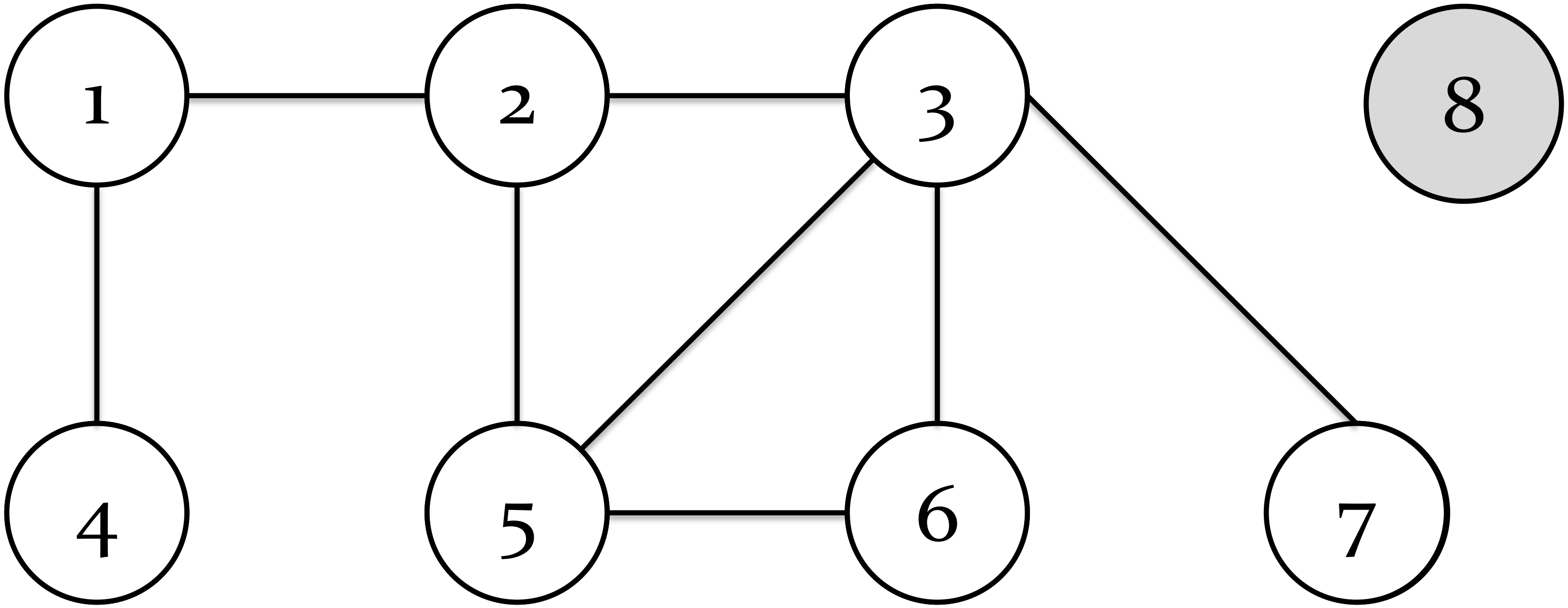
Layering algorithm

Example:

all vertices of unit weight: $w(v) = 1$

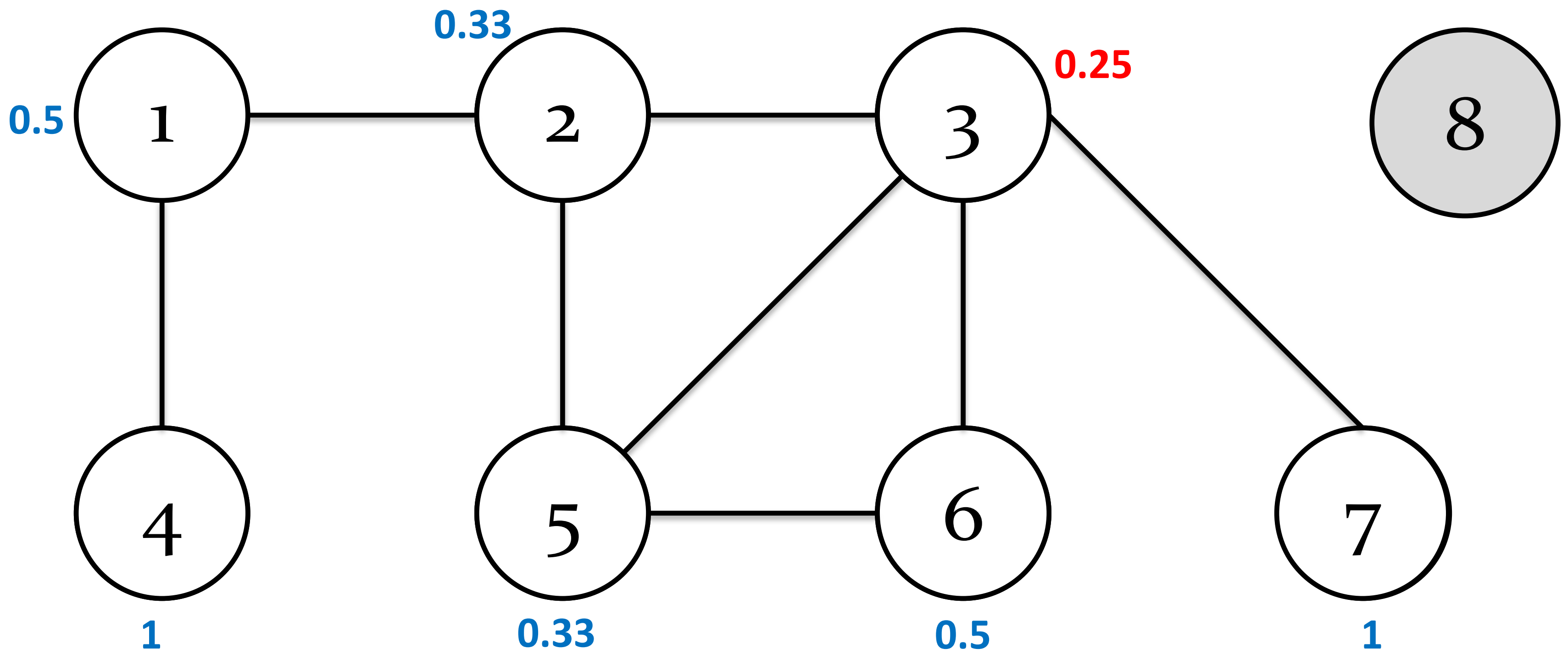


Iteration 0



$D_0 = \{\, 8 \,\}$

Iteration 0

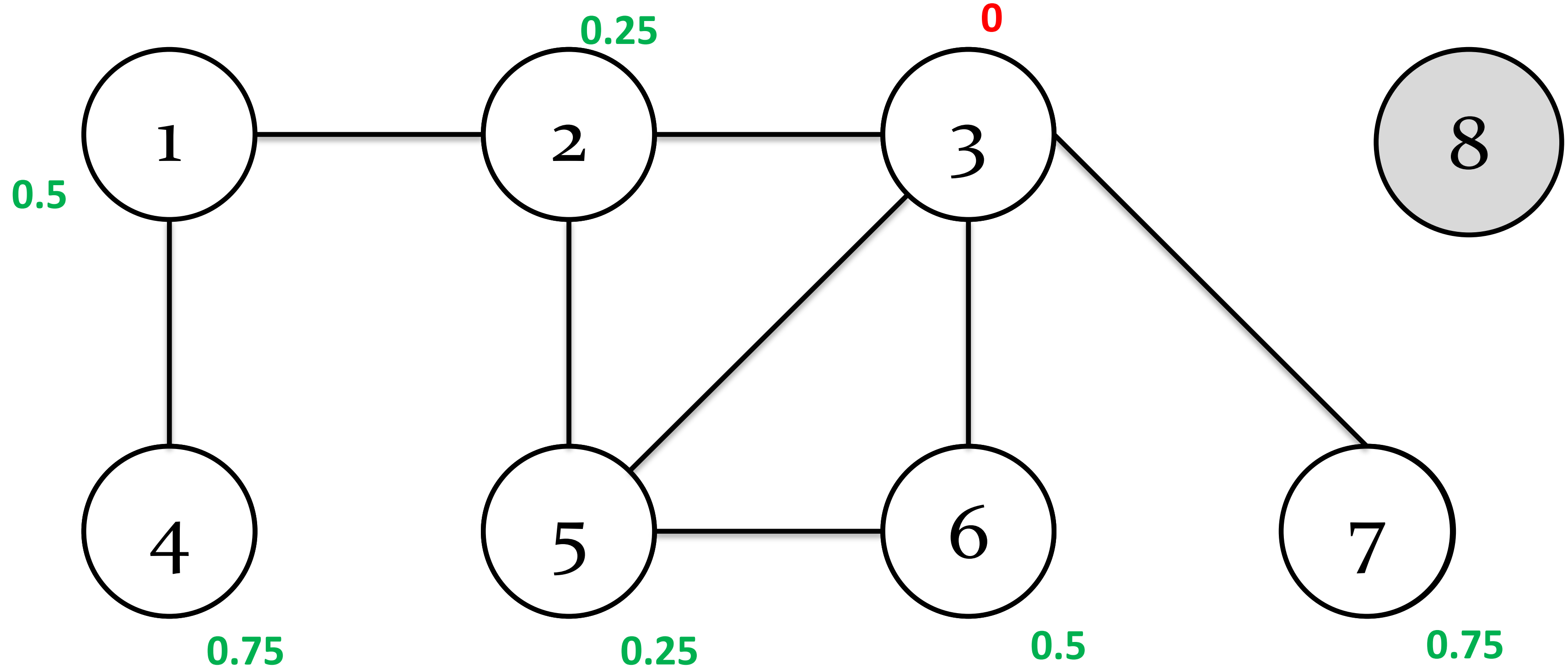


$D_0 = \{ 8 \}$

compute c = $min\{w(v)/deg(v)\}$ **= 0.25**

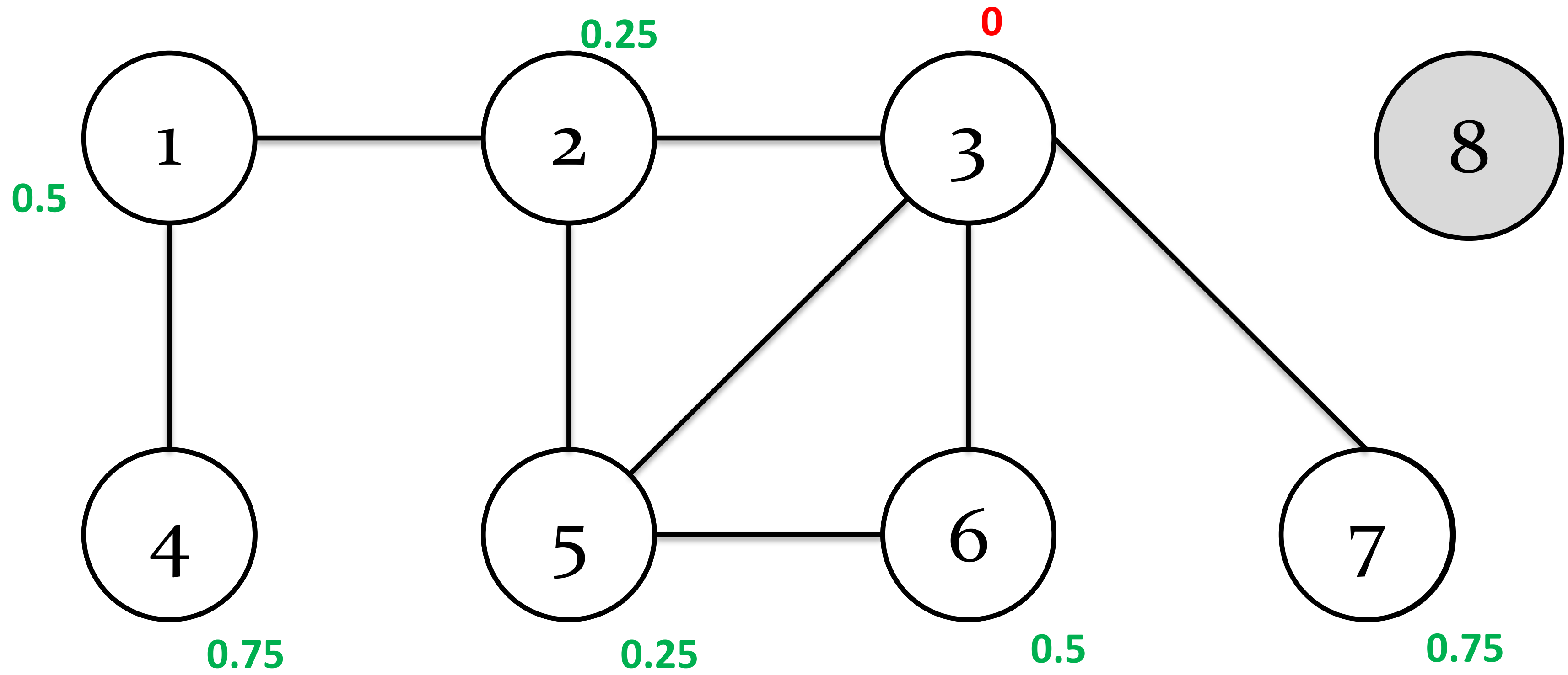
degree-weighted function: $t(v) = c \cdot \deg(v) = \mathbf{0.25} \cdot \boldsymbol{deg(v)}$

Iteration 0



$D_0 = \{ 8 \}$
compute c = $min\{w(v)/deg(v)\}$ **= 0.25**
degree-weighted function: $t(v) = c \cdot \text{deg(v)} = 0.25 \cdot \boldsymbol{deg(v)}$
compute residual weight: w'(v) = w(v) – t(v)

Iteration 0



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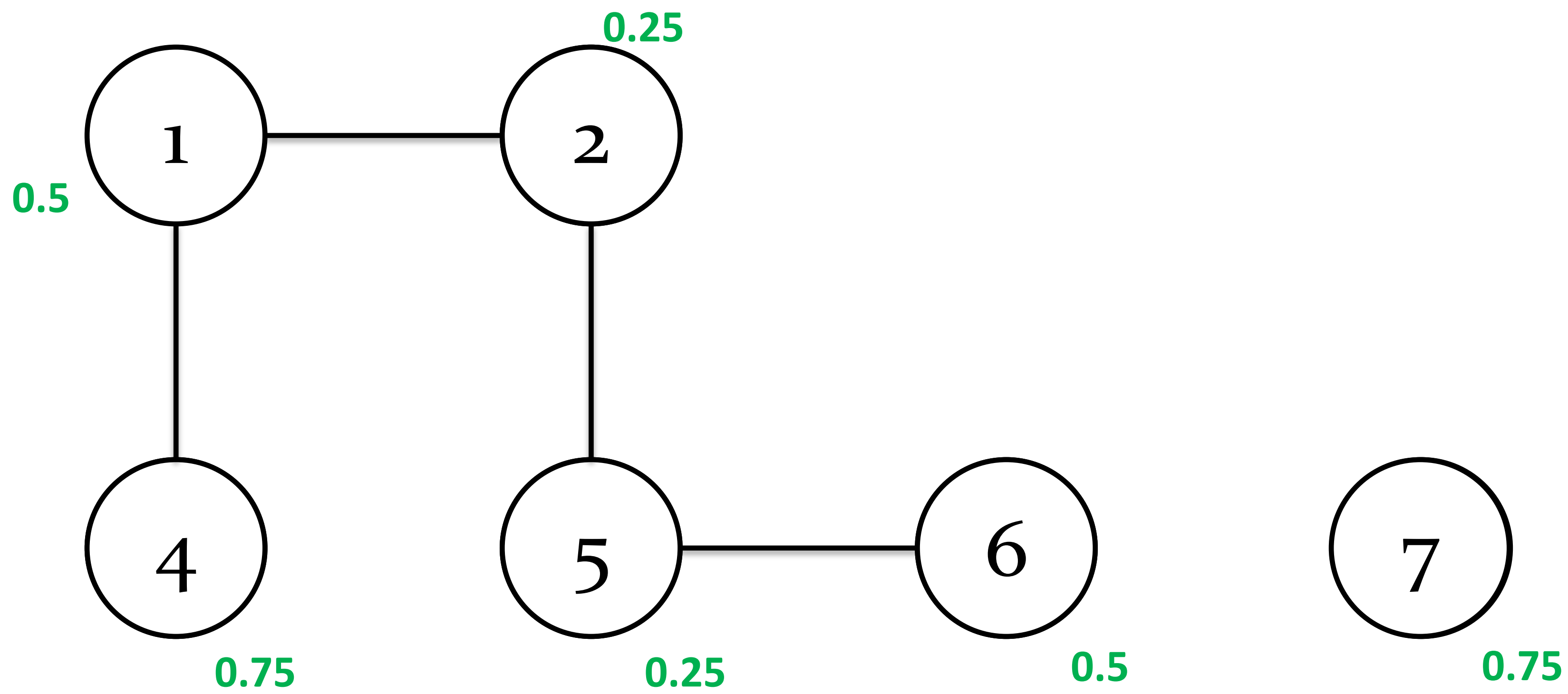
Iteration 0



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compute c = $min\{w(v)/deg(v)\}$ **= 0.25**

degree-weighted function: $t(v) = c \cdot \mathrm{deg(v)} = \mathbf{0.25} \cdot \boldsymbol{deg(v)}$

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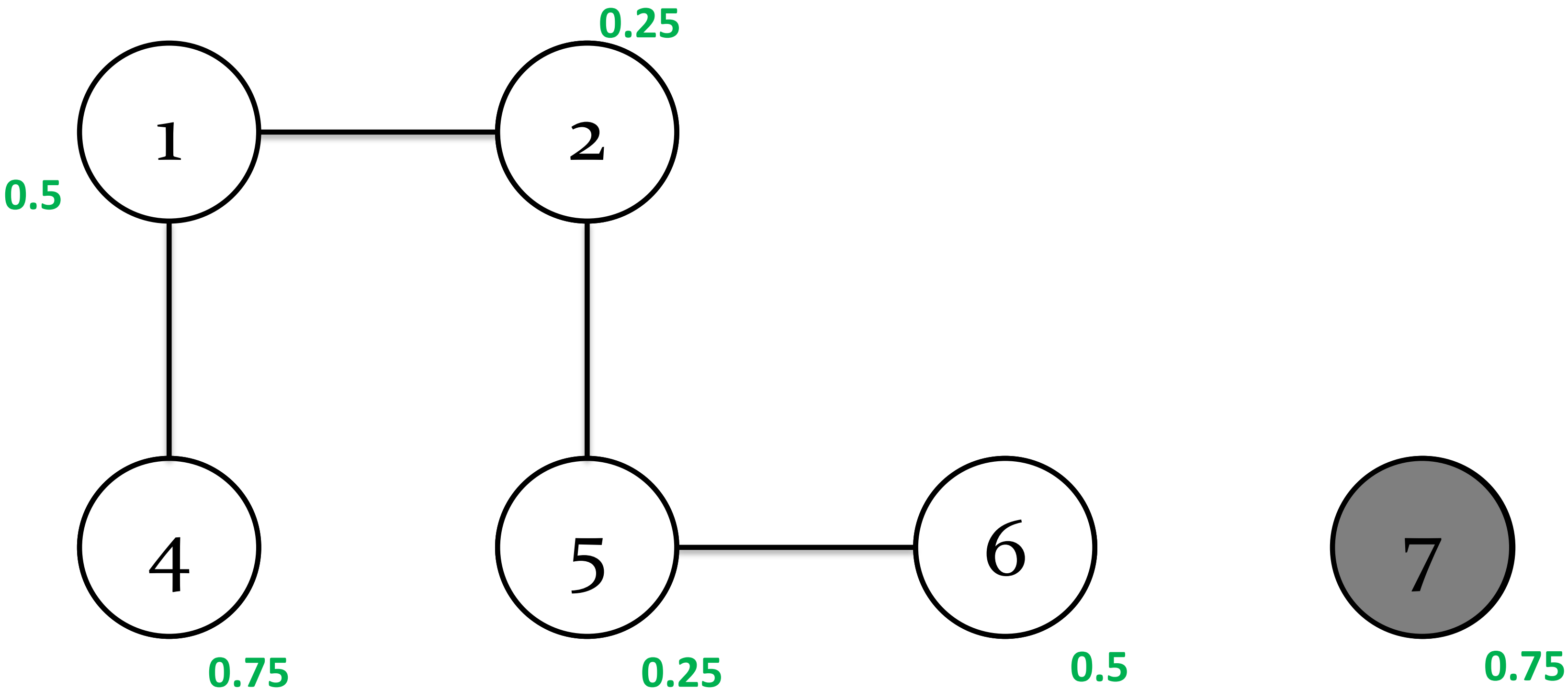
$W_0 = \{ 3 \}$

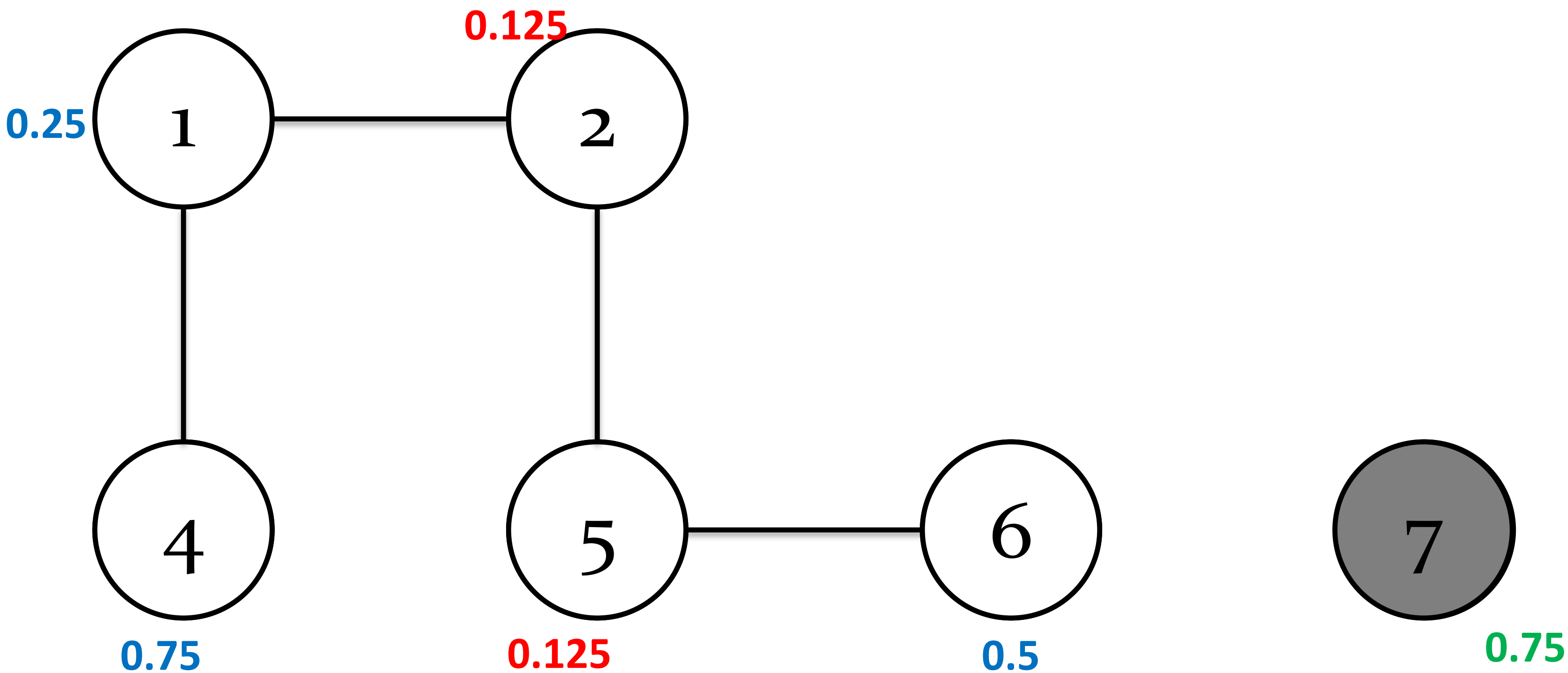
remove D_0 and W_0

Iteration 1



$D_1 = \{ 7 \}$

Iteration 1

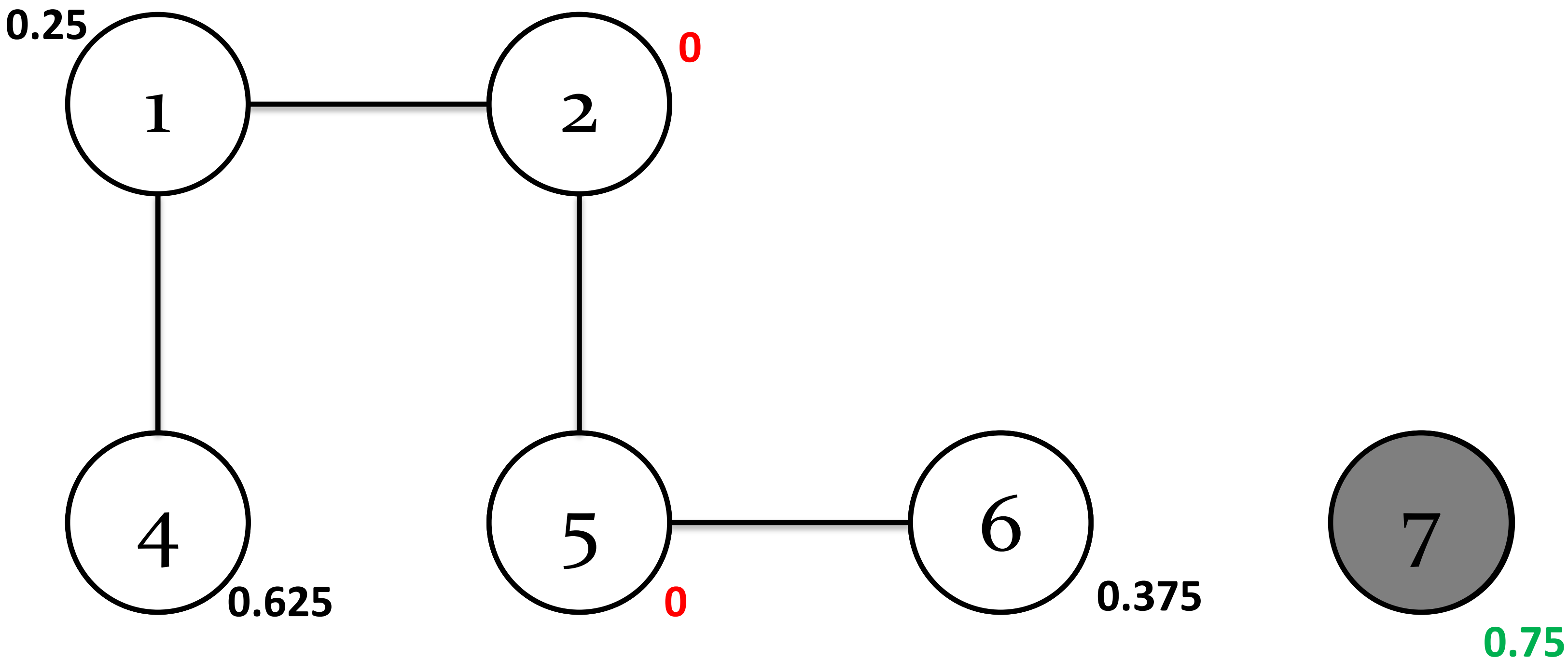


$D_1 = \{\ 7\ \}$

compute c = $min\{w(v)/deg(v)\}$ **= 0.125**

degree-weighted function: $t(v) = c \cdot \deg(v) = \mathbf{0.125} \cdot \boldsymbol{deg(v)}$

Iteration 1



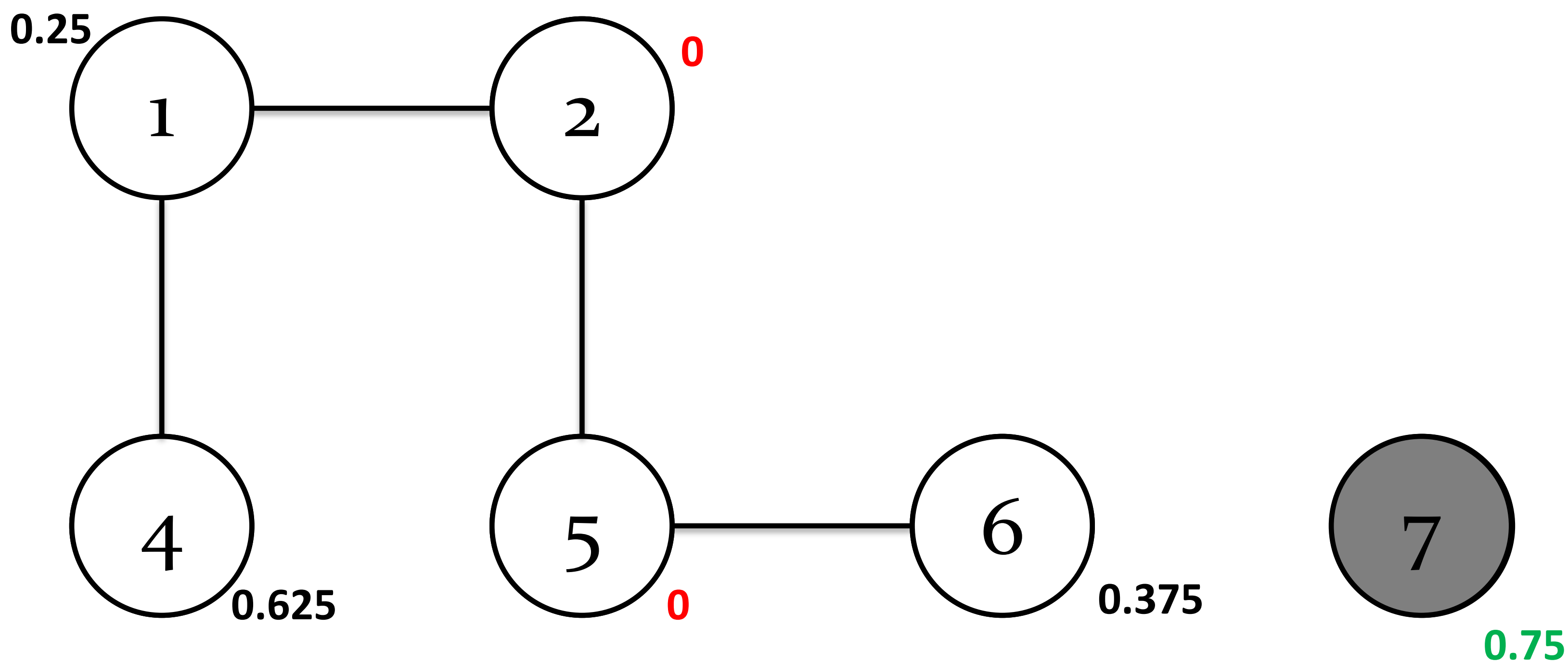
$D_1 = \{\ 7\ \}$

compute c = $min\{w(v)/deg(v)\}$ **= 0.125**

degree-weighted function: $t(v) = c \cdot \deg(v) = \mathbf{0.125} \cdot \boldsymbol{deg}(\boldsymbol{v})$

compute residual weight: w'(v) = w(v) – t(v)

Iteration 1



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compute c = $min\{w(v)/deg(v)\}$ **= 0.125**
degree-weighted function: $t(v) = c \cdot \deg(v) = \mathbf{0.125} \cdot \boldsymbol{deg}(\boldsymbol{v})$
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$W_1 = \{ 2, 5 \}$

Iteration 1

$D_1 = \{ 7 \}$

compute c = $min\{w(v)/deg(v)\}$ **= 0.125**

degree-weighted function: $t(v) = c \cdot \text{deg}(v) = \mathbf{0.125} \cdot \boldsymbol{deg}(\boldsymbol{v})$

compute residual weight: w'(v) = w(v) – t(v)

$W_1 = \{ 2, 5 \}$

remove D_1 and W_1

Iteration 2

$D_2 = \{\ \color{red}{6}\ \}$

Iteration 2

$D_2 = \{\ 6\ \}$

compute c = $min\{w(v)/deg(v)\}$ **= 0.25**

degree-weighted function: $t(v) = c \cdot \text{deg(v)} = 0.25 \cdot \boldsymbol{deg(v)}$

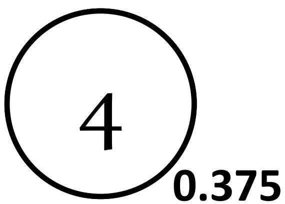
Iteration 2

$D_2 = \{\ 6\ \}$
compute c = $min\{w(v)/deg(v)\}$ **= 0.25**
degree-weighted function: $t(v) = c \cdot \deg(v) = \mathbf{0.25} \cdot \boldsymbol{deg(v)}$
compute residual weight: w'(v) = w(v) – t(v)

Iteration 2

$D_2 = \{ 6 \}$
compute c = $min\{w(v)/deg(v)\}$ **= 0.25**
degree-weighted function: $t(v) = c \cdot \deg(v) = \mathbf{0.25} \cdot \boldsymbol{deg}(\boldsymbol{v})$
compute residual weight: w'(v) = w(v) – t(v)
$W_2 = \{ 1 \}$

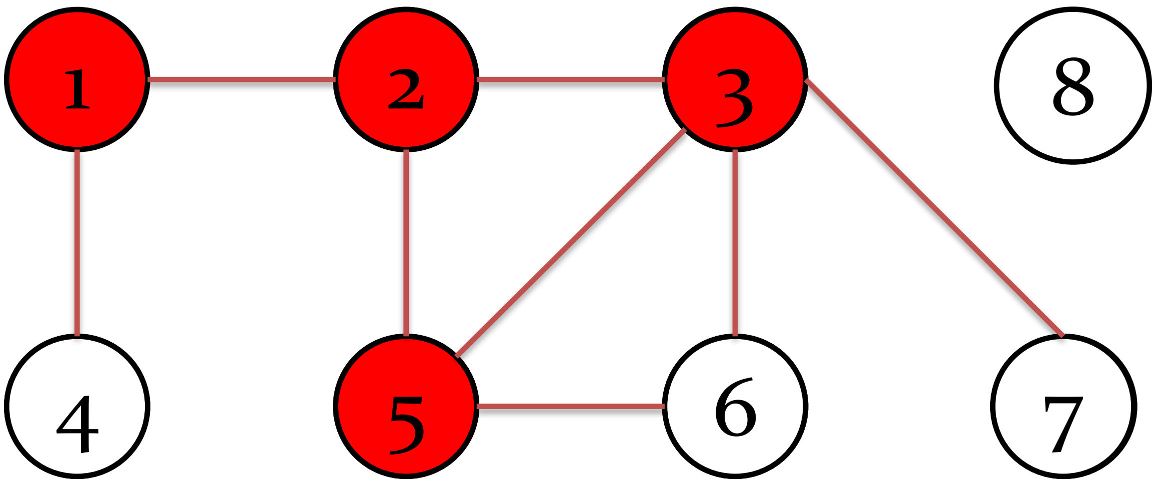
Iteration 2



$D_2 = \{ 6 \}$
compute c = $min\{w(v)/deg(v)\}$ **= 0.25**
degree-weighted function: $t(v) = c \cdot \mathrm{deg(v)} = \mathbf{0.25} \cdot \boldsymbol{deg(v)}$
compute residual weight: w'(v) = w(v) – t(v)
$W_2 = \{ 1 \}$
remove D_2 and W_2

Layering algorithm

all vertices of unit weight: w(v) = 1

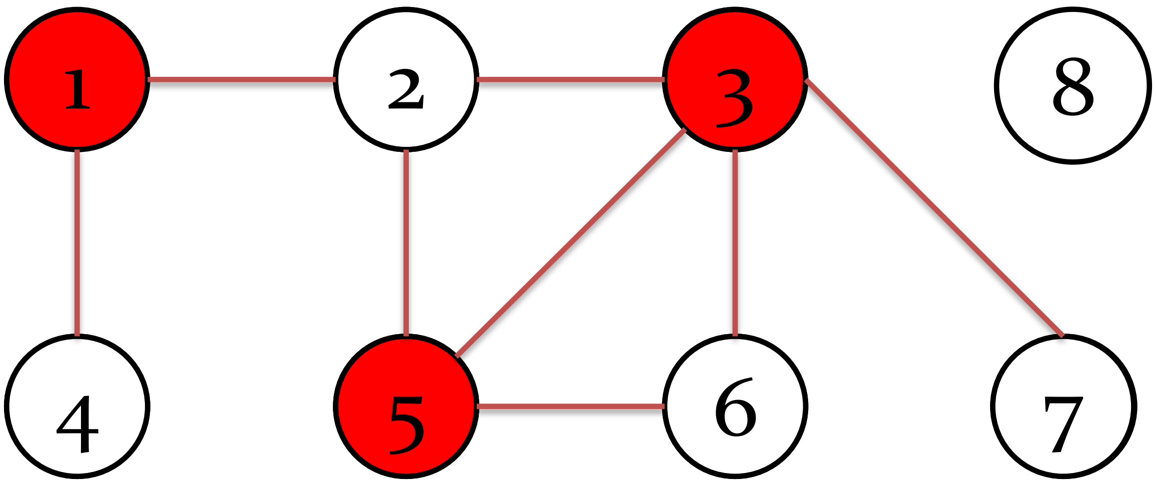


Vertex cover $\boldsymbol{C} = \boldsymbol{W_0} \cup \boldsymbol{W_1} \cup \boldsymbol{W_2} = \{\mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{5}\}$

Total cost: $\mathbf{w}(\boldsymbol{C}) = \mathbf{4}$

Layering algorithm

all vertices of unit weight: w(v) = 1

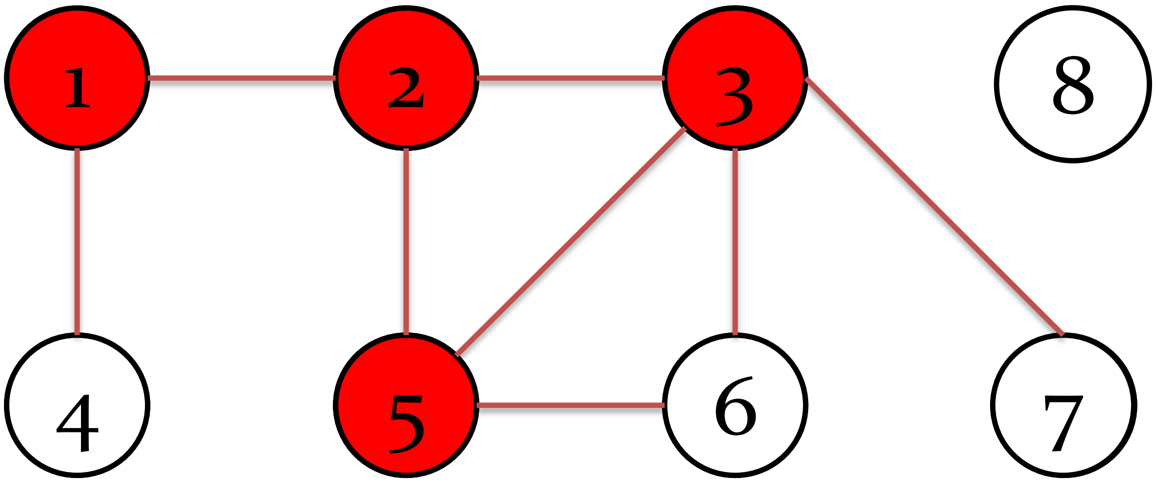


Optimal vertex cover $\boldsymbol{C}^{*} = \{\mathbf{1}, \mathbf{3}, \mathbf{5}\}$

Optimal cost: $\boldsymbol{OPT} = \mathbf{w}(\boldsymbol{C}^{*}) = \mathbf{3}$

Layering algorithm

all vertices of unit weight: w(v) = 1



Vertex cover $\boldsymbol{C} = \boldsymbol{W_0} \cup \boldsymbol{W_1} \cup \boldsymbol{W_2} = \{\mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{5}\}$
Total cost: $\mathbf{w}(\boldsymbol{C}) = \mathbf{4}$

Optimal cost: $\boldsymbol{OPT} = \mathbf{w}(\boldsymbol{C}^*) = \mathbf{3}$

$\mathbf{w}(\boldsymbol{C}) < \mathbf{2 \cdot OPT}$

Approximation factor

The layering algorithm for vertex cover problem (assuming arbitrary vertex weights) achieves an approximation guarantee of factor **2**.

Approximation factor

The layering algorithm for vertex cover problem (assuming arbitrary vertex weights) achieves an approximation guarantee of factor **2**.

We need to prove:

1) set C is a vertex cover for G
2) $w(C) \leq 2 \cdot OPT$

Proof

1) set C is a vertex cover for G

layering algorithm terminates until all nodes are zero degree.

all edges have been already covered.

Proof

2) $w(C) \leq 2 \cdot OPT$

a vertex $v \in C$, if $v \in W_j$,

its weight can be decomposed as $\mathbf{w}(\boldsymbol{v}) = \sum_{i \leq j} \boldsymbol{t_i}(\boldsymbol{v})$

a vertex $v \in V - C$, if $v \in D_j$,

a lower bound on its weight is given by $\mathbf{w}(\boldsymbol{v}) \geq \sum_{i \leq j} \boldsymbol{t_i}(\boldsymbol{v})$

Let C^* be the optimal vertex cover,
in each layer i, $\boldsymbol{C}^* \cap \boldsymbol{G_i}$ is a vertex cover for G_i

recall the lemma :
if $w: V \rightarrow Q^+$ is a degree-weighted function, then $w(V) \leq 2 \cdot OPT$
by lemma, $\mathbf{t_i}(\mathbf{C} \cap \mathbf{G_i}) \leq \mathbf{2} \cdot \mathbf{t_i}(\mathbf{C}^* \cap \mathbf{G_i})$

$$\mathbf{w}(\mathbf{C}) = \sum_{\mathbf{i=0}}^{\mathbf{k-1}} \mathbf{t_i}(\mathbf{C} \cap \mathbf{G_i}) \leq \mathbf{2} \cdot \sum_{\mathbf{i=0}}^{\mathbf{k-1}} \mathbf{t_i}(\mathbf{C}^* \cap \mathbf{G_i}) \leq \mathbf{2} \cdot \mathbf{w}(\mathbf{C}^*)$$

Summary

	Vertex Cover Problem	Set Cover Problem
Layering Algorithm	**factor = 2**	**factor = f**

Introduction to LP-Duality

Zhu Xiaolu

Introduction to LP-Duality

- Linear Programming (LP)
- LP-Duality
- Theorem

a) Weak duality theorem

b) LP-duality theorem

c) Complementary slackness conditions

Why use LP ?

obtaining approximation algorithms using LP

- Rounding Techniques

to solve the linear program and convert the fractional solution into an integral solution.

- Primal-dual Schema

to use the dual of the LP-relaxation in the design of the algorithm.

analyzing combinatorially obtained approximation algorithms

- LP-duality theory is useful
- using the method of dual fitting

What is LP ?

Linear programming: the problem of optimizing (i.e., minimizing or maximizing) a linear function subject to linear inequality constraints.

$$\begin{aligned} \text{Minimize} \quad & 7X_1 + X_2 + 5X_3 \\ \text{Subject to} \quad & X_1 - X_2 + 3X_3 \geq 10 \\ & 5X_1 + 2X_2 - X_3 \geq 6 \\ & X_1, X_2, X_3 \geq 0 \end{aligned}$$

Objective function: The linear function that we want to optimize.

Feasible solution: An assignment of values to the variables that satisfies the inequalities. E.g. X=(2,1,3)

Cost: The value that the objective function gives to an assignment. E.g. $7 \cdot 2 + 1 + 5 \cdot 3$=30

What is LP-Duality ?

Minimize $7X_1 + X_2 + 5X_3$

Subject to $X_1 - X_2 + 3X_3 \geq 10 \longrightarrow a \geq b$

$5X_1 + 2X_2 - X_3 \geq 6 \longrightarrow c \geq d$

$X_1, X_2, X_3 \geq 0$

Abstract the constraints:

$a \geq b$

$c \geq d$

Find multipliers: $y_1, y_2 \geq 0$

$y_1 a \geq y_1 b$

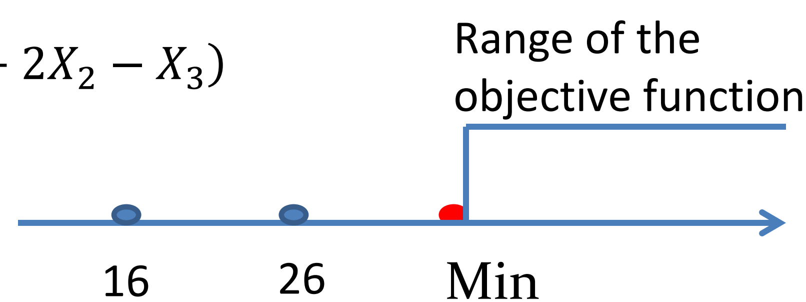
$y_2 c \geq y_2 d$

Objective function $\geq y_1 a + y_2 c \geq \boxed{y_1 b + y_2 d}$ (1)

As large as possible

$$7X_1 + X_2 + 5X_3 \geq \mathbf{1} \times (X_1 - X_2 + 3X_3) + \mathbf{1} \times (5X_1 + 2X_2 - X_3) = 6X_1 + X_2 + 2X_3 = 1 \times 10 + 1 \times 6 \geq 16$$

$$7X_1 + X_2 + 5X_3 \geq \mathbf{2} \times (X_1 - X_2 + 3X_3) + \mathbf{1} \times (5X_1 + 2X_2 - X_3) = 7X_1 + 5X_3 = 2 \times 10 + 1 \times 6 \geq 26$$



What is LP-Duality ?

The primal program

Minimize $7X_1 + X_2 + 5X_3$
Subject to $X_1 - X_2 + 3X_3 \geq 10$
$5X_1 + 2X_2 - X_3 \geq 6$
$X_1, X_2, X_3 \geq 0$

The dual program

Maximize $10Y_1 + 6Y_2$
Subject to $Y_1 + 5Y_2 \leq 7$
$-Y_1 + 2Y_2 \leq 1$
$3Y_1 - Y_2 \leq 5$
$Y_1, Y_2 \geq 0$

Minimize $\sum_{j=1}^{n} c_j x_j$
Subject to $\sum_{j=1}^{n} a_{ij} x_j \geq b_i$,
$i = 1, \cdots, m$
$x_j \geq 0$, $j = 1, \cdots, n$
a_{ij}, b_i , c_j are given rational numbers

Maximize $\sum_{i=1}^{m} b_i y_i$
Subject to $\sum_{i=1}^{m} a_{ij} y_i \leq c_j$,
$j = 1, \cdots, n$
$y_i \geq 0$, $i = 1, \cdots, m$
a_{ij}, b_i , c_j are given rational numbers

Minimize $c_1X_1 + \cdots + c_nX_n$
Subject to $a_{1,1}X_1 + \cdots + a_{1,n}X_n \geq b_1$
$\cdots$
$a_{m,1}X_1 + \cdots + a_{m,n}X_n \geq b_m$
$X_1, \cdots, X_n \geq 0$

Minimize $7X_1 + X_2 + 5X_3$
Subject to $X_1 - X_2 + 3X_3 \geq 10$
$5X_1 + 2X_2 - X_3 \geq 6$
$X_1, X_2, X_3 \geq 0$

$$Y_1(a_{1,1}X_1 + \cdots + a_{1,n}X_n) + \cdots + Y_m(a_{m,1}X_1 + \cdots + a_{m,n}X_n) \geq Y_1b_1 + \cdots + Y_mb_m \quad (1)$$

$$(a_{1,1}Y_1 + \cdots + a_{m,1}Y_m)X_1 + \cdots + (a_{1,n}Y_1 + \cdots + a_{m,n}Y_m)\,X_n \geq Y_1b_1 + \cdots + Y_mb_m \quad (2)$$

Assume:

$$a_{1,1}Y_1 + \cdots + a_{m,1}Y_m \leq c_1$$
$$\cdots$$
$$a_{1,n}Y_1 + \cdots + a_{m,n}Y_m \leq c_n$$

$$c_1X_1 + \cdots + c_nX_n \geq (a_{1,1}Y_1 + \cdots + a_{m,1}Y_m)X_1 + \cdots + (a_{1,n}Y_1 + \cdots + a_{m,n}Y_m)\,X_n \quad (3)$$
$$\geq b_1Y_1 + \cdots + b_mY_m$$

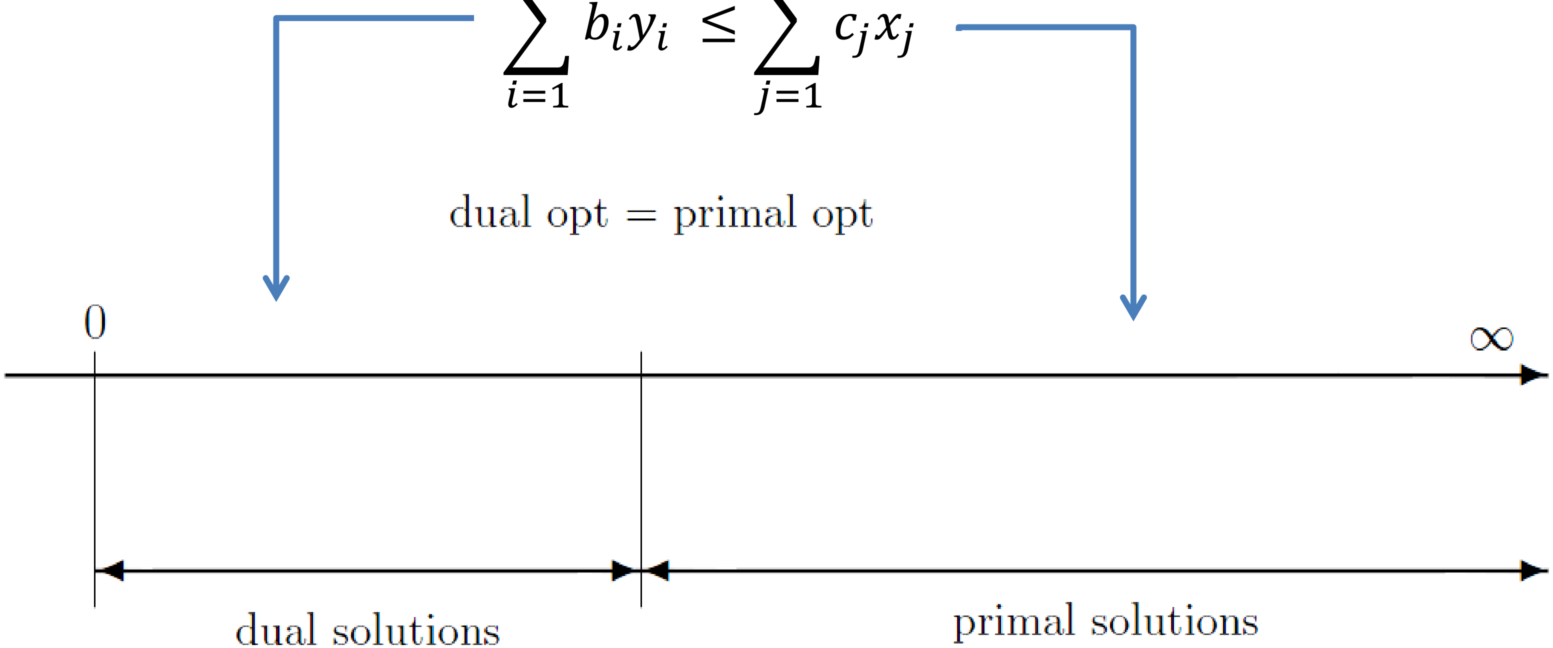
Maximize $b_1Y_1 + \cdots + b_mY_m$
Subject to $a_{1,1}Y_1 + \cdots + a_{m,1}Y_m \leq c_1$
$\cdots$
$a_{1,n}Y_1 + \cdots + a_{m,n}Y_m \leq c_n$
$Y_1, \cdots, Y_m \geq 0$

Maximize $10Y_1 + 6Y_2$
Subject to $Y_1 + 5Y_2 \leq 7$
$-Y_1 + 2Y_2 \leq 1$
$3Y_1 - Y_2 \leq 5$
$Y_1, Y_2 \geq 0$

Weak duality theorem

If $X=(X_1,\cdots,X_n)$ and $Y=(Y_1,\cdots,Y_m)$ are feasible solutions for the primal and dual program, respectively, then

$$\sum_{i=1}^{m} b_i y_i \leq \sum_{j=1}^{n} c_j x_j$$



Weak duality theorem - proof

Minimize $\sum_{j=1}^{n} c_j x_j$ Subject to $\sum_{j=1}^{n} a_{ij} x_j \ge b_i$, $i = 1, \cdots, m$ $x_j \ge 0$, $j = 1, \cdots, n$ a_{ij}, b_i , c_j are given rational numbers	Maximize $\sum_{i=1}^{m} b_i y_i$ Subject to $\sum_{i=1}^{m} a_{ij} y_i \le c_j$, $j = 1, \cdots, n$ $y_i \ge 0$, $i = 1, \cdots, m$ a_{ij}, b_i , c_j are given rational numbers

Since y is dual feasible and x_j's are nonnegative,

$$\sum_{j=1}^{n} c_j x_j \ge \sum_{j=1}^{n} \left(\sum_{i=1}^{m} a_{ij} y_i \right) x_j$$

Since x is primal feasible and y_j's are nonnegative,

$$\sum_{i=1}^{m} \left(\sum_{j=1}^{n} a_{ij} x_j \right) y_i \ge \sum_{i=1}^{m} b_i y_i$$

Obviously,

$$\sum_{j=1}^{n} \left(\sum_{i=1}^{m} a_{ij} y_i \right) x_j = \sum_{i=1}^{m} \left(\sum_{j=1}^{n} a_{ij} x_j \right) y_i$$

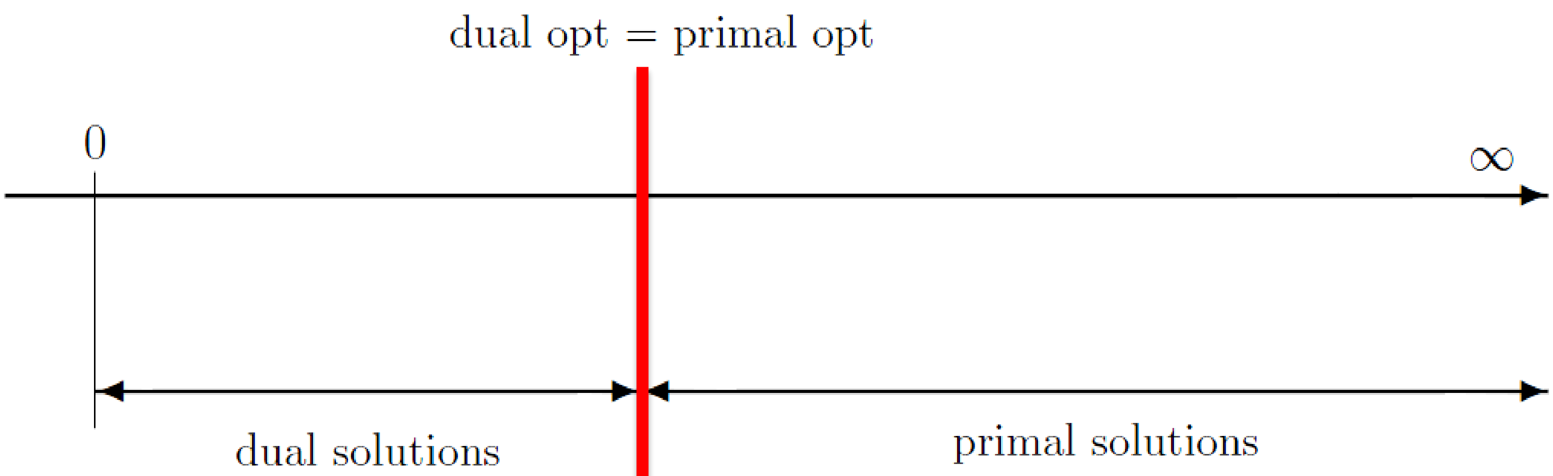
So, we proof that

$$\sum_{j=1}^{n} c_j x_j \ge \sum_{i=1}^{m} b_i y_i$$

LP-duality theorem

If $X^* = (X_1^*, \cdots, X_n^*)$ and $Y^* = (Y_1^*, \cdots, Y_m^*)$ are optimal solutions for the primal and dual programs, respectively, then

$$\sum_{j=1}^{n} c_j x_j^* = \sum_{i=1}^{m} b_i y_i^*$$



Complementary slackness conditions

Let X and Y be primal and dual feasible solutions, respectively. Then, X and Y are both optimal iff all of the following conditions are satisfied:

- **Primal complementary slackness conditions**

For each $1 \le j \le n$: either $x_j = 0$ or $\sum_{i=1}^{m} a_{ij} y_i = c_j$;

And

- **Dual complementary slackness conditions**

For each $1 \le i \le m$: either $y_i = 0$ or $\sum_{j=1}^{n} a_{ij} x_j = b_i$.

Complementary slackness conditions -proof

Proof how these two conditions comes up
From the proof of weak duality theorem:

$$\sum_{j=1}^{n} c_j x_j \geq \sum_{j=1}^{n} \left(\sum_{i=1}^{m} a_{ij} y_i \right) x_j \qquad (1)$$

$$\sum_{i=1}^{m} \left(\sum_{j=1}^{n} a_{ij} x_j \right) y_i \geq \sum_{i=1}^{m} b_i y_i \qquad (2)$$

By the LP-duality theorem, x and y are both optimal solutions iff:

$$\sum_{j=1}^{n} c_j x_j^* = \sum_{i=1}^{m} b_i y_i^*$$

These happens iff (1) and (2) hold with equality:

$$\sum_{j=1}^{n} c_j x_j^* = \sum_{j=1}^{n} \left(\sum_{i=1}^{m} a_{ij} y_i \right) x_j^* = \sum_{i=1}^{m} \left(\sum_{j=1}^{n} a_{ij} x_j \right) y_i^* = \sum_{i=1}^{m} b_i y_i^*$$

$$\sum_{j=1}^{n} c_j x_j^* = \sum_{j=1}^{n} \left(\sum_{i=1}^{m} a_{ij} y_i\right) x_j^* = \sum_{i=1}^{m} \left(\sum_{j=1}^{n} a_{ij} x_j\right) y_i^* = \sum_{i=1}^{m} b_i y_i^*$$

$$\sum_{j=1}^{n} \left(c_j - \left(\sum_{i=1}^{m} a_{ij} y_i\right)\right) x_j^* = 0$$

Same Process

Dual complementary slackness conditions

For each $1 \leq j \leq n$

If $x_j^* > 0$ then $c_j - \left(\sum_{i=1}^{m} a_{ij} y_i\right) = 0$

If $c_j - \left(\sum_{i=1}^{m} a_{ij} y_i\right) > 0$ then $x_j^* = 0$

Primal complementary slackness conditions

For each $1 \leq j \leq n$: either $x_j = 0$ or $\sum_{i=1}^{m} a_{ij} y_i = c_j$

Set Cover via Dual Fitting

Wang Zixiao

Set Cover via Dual Fitting

- Dual Fitting: help analyze combinatorial algorithms using LP-duality theory

- Analysis of the greedy algorithm for the set cover problem

- Give a lower bound

Dual Fitting

- Minimization problem: combinatorial algorithm
- Linear programming relaxation, Dual
- Primal is ***fully paid for*** by the dual
- Infeasible → Feasible (shrunk with factor)
- Factor → Approximation guarantee

Formulation

U: universe

S: $\{S_1, S_2, \ldots, S_k\}$

C: ***S*** —> Q^+

Goal:sub-collection with minimum cost

$$\downarrow$$

$$\text{minimize} \quad \sum_{S \in \mathcal{S}} c(S) x_S, \quad x_S \in \{0, \ 1\}$$

$$\text{subject to} \quad \forall e \in U, \sum_{S : e \in S} x_S \geq 1$$

LP-Relaxation

- Motivation: NP-hard → Polynomial
- Integer program → Fractional program
- Letting x_S: $0 \leq x_S$

$$\textit{minimize} \quad \sum_{S \in \mathcal{S}} c(S) x_S$$

$$\textit{subject to} \quad \forall e \in U, \sum_{S : e \in S} x_S \geq 1$$

$$x_S \geq 0, \qquad S \in \mathcal{S}$$

LP Dual Problem

$$\textit{minimize} \quad \sum_{S \in \mathbf{S}} c(S) x_S$$

$$\textit{subject to} \quad \forall e \in U, \sum_{S: e \in S} x_S \geq 1$$

$$x_S \geq 0, \qquad S \in \mathbf{S}$$

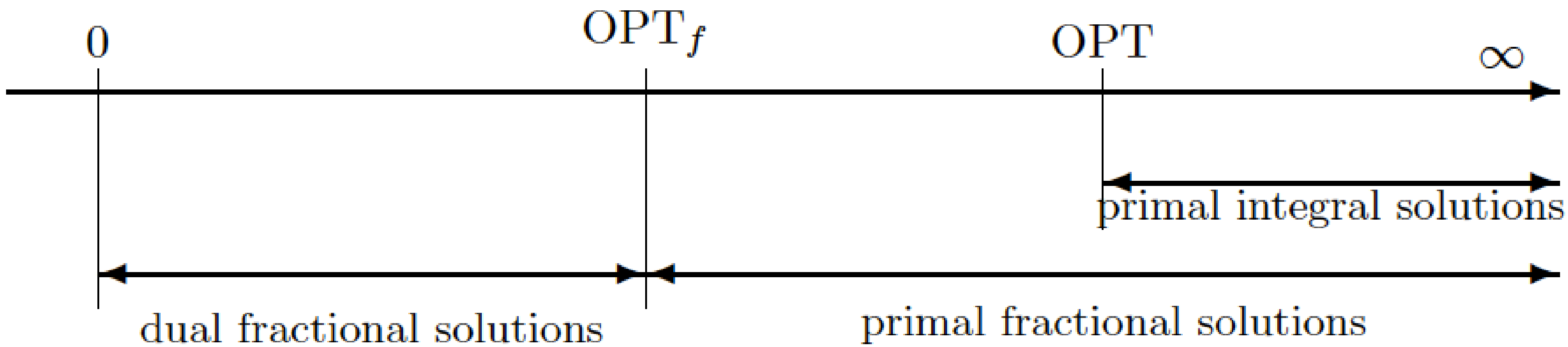
$$\downarrow$$

$$\textit{maximize} \quad \sum_{e \in U} y_e$$

$$\textit{subject to} \quad \forall S \in \mathbf{S}, \sum_{e:\ e \in S} y_e \leq c(S)$$

$$y_e \geq 0, \qquad e \in U$$

LP Dual Problem



OPT: cost of optimal integral set cover
OPT$_f$: cost of optimal fractional set cover

Solution

y_e = price(e)?

Not feasible!

U = {1, 2, 3}
S = {{1, 2, 3}, {1, 3}}
C({1, 2, 3}) = 3
C({1, 3}) = 1

- Iteration 1: {1, 3} chosen
  price(1) = price(3) = 0.5
- Iteration 2: {1, 2, 3} chosen
  price(2) = 3

price(1) + price(2) + price(3) = 4 > C({1, 2, 3})

violation

$$\text{maximize} \quad \sum_{e \in U} y_e$$

$$\text{subject to} \quad \forall S \in \mathbf{S}, \sum_{e:\ e \in S} y_e \leq c(S)$$

$$y_e \geq 0, \qquad e \in U$$

Solution

$$y_e = \frac{price(e)}{H_n}$$

*The vector **y** defined above is a feasible solution for the dual program*

$\Updownarrow$

There is no set S in **S** overpacked

Proof

Consider a set S∈**S**
| S | = k

Number the elements
in the order of being covered:
$e_1, e_2, \ldots, e_k$

Consider the iteration for e_i:
$e_1, e_2, \ldots, \underline{e_i}, \ldots, e_k$

At least k-i+1 elements
uncovered in S

Select S:

$$price(e_i) \le \frac{C(S)}{k-i+1}$$

Select S' :

$$price(e_i) < \frac{C(S)}{k-i+1}$$

$$price(e_i) \le \frac{C(S)}{k-i+1}$$

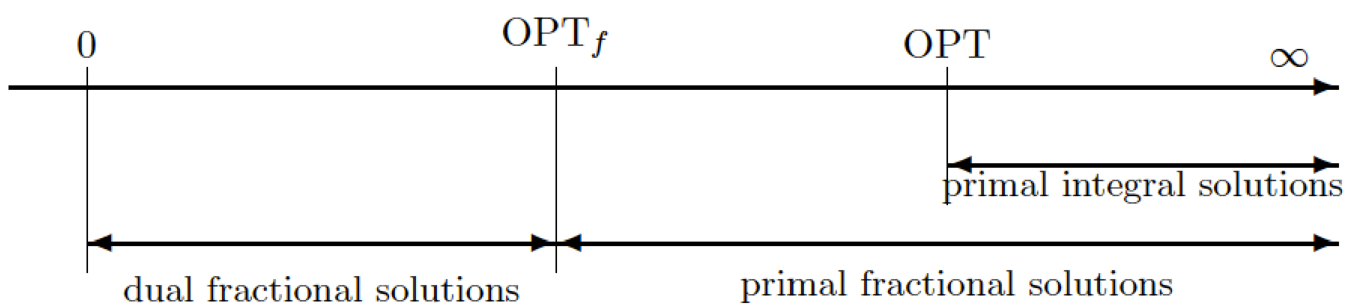
$$y_{e_i} = \frac{price(e_i)}{H_n} \le \frac{C(S)}{H_n} * \frac{1}{k-i+1}$$

$$\sum_{i=1}^{k} y_{e_i} \le \frac{C(S)}{H_n} * \left(\frac{1}{k} + \frac{1}{k-1} + \cdots + \frac{1}{1}\right) = \frac{H_k}{H_n} * C(S) \le C(S)$$

The approximation of guarantee of the greedy set cover algorithm is H_n

Proof: The cost of the set cover picked is:

$$\sum_{e \in U} price(e) = H_n(\sum_{e \in U} y_e) \leq H_n * OPT_f \leq H_n OPT$$



Constrained Set Multicover

- Constrained Set Multicover: Each element e has to be covered r_e times

- Greedy algorithm: Pick the most cost-effective set until all elements have been covered by required times

Analysis of the greedy algorithm

IP formulation:

- minimize $\sum_{S \in \mathcal{S}} c(S) \cdot x_S$
- subject to:
  - $\sum_{S: e \in S} x_S \geq r_e, \; e \in \mathcal{U}$
  - $x_S \in \{0, 1\}, \; S \in \mathcal{S}$

LP relaxation:

- minimize $\sum_{S \in \mathcal{S}} c(S) \cdot x_S$
- subject to:
  - $\sum_{S: e \in S} x_S \geq r_e, \; e \in \mathcal{U}$
  - $-x_S \geq -1, \; S \in \mathcal{S}$
  - $x_S \geq 0, \; S \in \mathcal{S}$

Dual program:

- maximize $\sum_{e \in \mathcal{U}} r_e \cdot y_e - \sum_{S \in \mathcal{S}} z_S$
- subject to:
  - $\sum_{e \in S} y_e - z_S \leq c(S), \; S \in \mathcal{S}$
  - $y_e \geq 0, \; e \in \mathcal{U}$
  - $z_S \geq 0, \; S \in \mathcal{S}$

Analysis of the greedy algorithm

Consider the objective function value D of the dual solution (α, β), where $\alpha_e = \text{price}(e, r_e)$ for each $e \in \mathcal{U}$ and

$$\beta_S = \sum_{e \text{ covered by } S} \left(\text{price}(e, r_e) - \text{price}(e, j_e)\right),$$

for each $S \in \mathcal{S}$ that was picked by the algorithm and $\beta_S = 0$ otherwise.

$\mathrm{D} = \sum_{e \in \mathcal{U}} r_e \cdot \text{price}(e, r_e) -$
$\sum_{e \in \mathcal{U}} \left(r_e \cdot \text{price}(e, r_e) - \sum_{j=1}^{r_e} \text{price}(e, j) \right) = \mathrm{SOL}$

Analysis of the greedy alrogithm

Lemma 2 *The pair* $(\mathbf{y}, \mathbf{z})$ *where* $y_e = \frac{\alpha_e}{H_n}$ *and* $z_S = \frac{\beta_S}{H_n}$ *is a feasible solution for the dual program.*

Therefore, $\frac{\mathrm{D}}{H_n} \leq \mathrm{OPT_f} \Rightarrow \mathrm{SOL} \leq H_n \cdot \mathrm{OPT}$ which implies an approximation factor of H_n for the previous algorithm.

Analysis of the greedy algorithm

Proof

Consider $S \in \mathcal{S}$ and its elements $e_1, e_2, \ldots, e_i, \ldots, e_k$ in the order in which their requirements are covered by the algorithm.

Case 1: S is not picked

Just before the last copy of e_i is covered, S contains at least $k-i+1$ alive elements. So, price$(e_i, r_{e_i}) \leq \frac{c(S)}{k-i+1}$. Moreover, since $z_S = 0$ we get:

$$\sum_{e \in S} y_e - z_S \leq \frac{1}{H_n} \cdot \sum_{i=1}^{k} \frac{c(S)}{k-i+1} \leq c(S)$$

Analysis of the greedy algorithm

Case 2: S **is picked**

Assume that just before S is picked, k' of its elements are completely covered.

$$\sum_{e \in S} y_e - z_S = \frac{1}{H_n} \cdot \left(\sum_{i=1}^{k} \text{price}\left(e_i, r_{e_i}\right) - \sum_{i=k'+1}^{k} \left(\text{price}\left(e_i, r_{e_i}\right) - \text{price}\left(e_i, j_i\right)\right) \right)$$

$$= \frac{1}{H_n} \cdot \left(\sum_{i=1}^{k'} \text{price}\left(e_i, r_{e_i}\right) + \sum_{i=k'+1}^{k} \text{price}\left(e_i, j_i\right) \right) \leq c\left(S\right)$$

Rounding Applied to LP to solve Set Cover

Jiao Qing

Why rounding?

- Previous slides show LP-relaxation for the set cover problem, but for set cover real applications, they usually need integral solution.
- Next section will introduce two rounding algorithms and their approximation factor to OPT_f and OPT.

Rounding Algorithms

- A simple rounding algorithm
- Randomized rounding

A simple rounding algorithm

- Algorithm Description
  1. Find an optimal solution to the LP-relaxation.
  2. Pick all sets S for which $x_s \geq \frac{1}{f}$ in this solution.
- Apparently, its algorithm complexity equals to LP problem.

A simple rounding algorithm

- Proving that it solves set cover problem.
  1. Remember the LP-relaxation:

     minimize $\sum_{s \in \boldsymbol{S}} c(s) x_s$

     subject to $\sum_{s: e \in s} x_s \geq 1, \quad e \in U$

     $x_s \geq 0, S \in \mathbf{S}$
  2. Let C be the collection of picked sets. For $\forall e \in U$, e is in at most f sets, one of these sets must has $x_s \geq \frac{1}{f}$, therefore, this simple algorithm at least solve the set cover problem.

A simple rounding algorithm

- Firstly we prove that it solves set cover problem.
- Then we estimate its approximation factor to OPT_f and OPT is f.

Approximation Factor

- Estimating its approximation factor
  1. The rounding process increases x_s by a factor of at most f, and further it reduces the number of sets.
  2. Thus it gives a desired approximation guarantee of f.

Randomized Rounding

- This rounding views the LP-relaxation coefficient x_s of sets as probabilities.
- Using probabilities theory it proves this rounding method has approximation factor $O(\log n)$.

Randomized Rounding

- Algorithm Description:
  1. Find an optimal solution to the LP-relaxation.
  2. Independently picks $v(\log n)$ subsets of full set S.
  3. Get these subsets' union C', check whether C' is a valid set cover and has cost $\leq OPT_f \times 4v \log n$. If not, repeat the step 2 and 3 again.
  4. The expected number of repetitions needed at most 2.

- Apparently, its algorithm complexity equals to LP problem.

Proof Approximation Factor

- Next,we prove its approximate factor is $O(\log n)$.
  1. For each set S, its probability of being picked is p_s(equals to x_s coefficient).
  2. Let C be one collection of sets picked. The expected cost of C is:

$$E[cost(C)] = \sum_{s \in \boldsymbol{s}} p_r[s\ is\ picked] \times c_s = \sum_{s \in \boldsymbol{s}} p_s \times c_s = OPT_f$$

Proof Approximation Factor

3. Let us compute the probability that an element a is covered by C.
   - Suppose that a occurs in k sets of S, with the probabilities associated with these sets be $p_1, \ldots p_k$. Because their sum greater than 1. Using elementary calculus, we get:

     $$p_r[a \text{ is covered by } C] \geq 1 - \left(1 - \frac{1}{k}\right)^k \geq 1 - \frac{1}{e}$$

     and thus,

     $$p_r[a \text{ is not covered by } C] \leq \frac{1}{e}$$

     where e is the base of natural logarithms.
   - Hence each element is covered with constant probability by C.

Proof Approximation Factor

4. And then considering their union C', we get:

$$p_r[a\ is\ not\ covered\ by\ c'] \leq \left(\frac{1}{e}\right)^{v(\log n)} \leq \frac{1}{4n}$$ With $\left(\frac{1}{e}\right)^{v \log n} \leq \frac{1}{4n}$

5. Summing over all elements $a \in U$, we get

$$p_r[c'\ is\ not\ a\ valid\ set\ cover\ ] = 1 - \left(1 - \frac{1}{4n}\right)^n \leq \frac{1}{4}$$

6. With:

$E[cost(C')] \leq \sum_i E[cost(c_i)]$=$OPT_f \times v \log n = OPT_f \times v \log n$

Applying Markov's Inequality with t=$OPT_f \times 4v \log n$, we get:

$$p_r\left[cost(C') \geq OPT_f \times 4v \log n\right] \leq \frac{1}{4}$$

Proof Approximation Factor

7. With

$$p_r[c' \, is \; not \; a \; valid \; set \; cover\,] \leq \frac{1}{4}$$

$$p_r\left[cost(C') \geq OPT_f \times 4v \log n\right] \leq \frac{1}{4}$$

Hence,

$$p_r\left[C' \, is \; a \; valid \; set \; cover \; and \; has \; cost \leq OPT_f \times 4v \log n\right] \geq \frac{3}{4} \times \frac{3}{4} \geq \frac{1}{2}$$

Approximation Factor

- The chance one could find a set cover and has a cost smaller than $O(\log n)OPT_f$ is bigger than 50%, and the expected number of iteration is two.
- Thus this randomized rounding algorithm provides a factor $O(\log n)$ approximation, with a high probability guarantee.

Summary

- Those two algorithm provide us with factor f and $O(\log n)$ approximation, which remind us the factor of greedy and layering algorithms'.
- Even through LP-relaxation, right now we could only find algorithm with factor of f and $O(\log n)$.

Set Cover via the Primal-dual Schema

Zhang Hao

Primal-dual Schema?

- A broad outline for the algorithm

  The details have to be designed individually to specific problems

- Good approximation factors and good running times

Primal-dual Program(Recall)

The primal program

Minimize $\sum_{j=1}^{n} c_j x_j$

Subject to $\sum_{j=1}^{n} a_{ij} x_j \geq b_i$,

$i = 1, \cdots, m$

$x_j \geq 0$, $j = 1, \cdots, n$

a_{ij}, b_i , c_j are given rational numbers

The dual program

Maximize $\sum_{i=1}^{m} b_i y_i$

Subject to $\sum_{i=1}^{m} a_{ij} y_i \leq c_j$,

$j = 1, \cdots, n$

$y_i \geq 0$, $i = 1, \cdots, m$

a_{ij}, b_i , c_j are given rational numbers

Standard Complementary slackness conditions (Recall)

Let **x** and **y** be primal and dual feasible solutions, respectively. Then, **x** and **y** are both optimal iff all of the following conditions are satisfied:

- **Primal complementary slackness conditions**

For each $1 \le j \le n$: either $x_j = 0$ or $\sum_{i=1}^{m} a_{ij} y_i = c_j$;
And

- **Dual complementary slackness conditions**

For each $1 \le i \le m$: either $y_i = 0$ or $\sum_{j=1}^{n} a_{ij} x_j = b_i$.

Relaxed Complementary slackness conditions

Let **x** and **y** be primal and dual feasible solutions, respectively.

- **Primal complementary slackness conditions**

For each $1 \leq j \leq n$: either $x_j = 0$ or $c_i/\alpha \leq \sum_{i=1}^{m} a_{ij}\, y_i \leq c_i$;

And

- **Dual complementary slackness conditions**

For each $1 \leq i \leq m$: either $y_i = 0$ or $b_i \leq \sum_{j=1}^{n} a_{ij} x_j \leq \beta b_i$.

*α, β => The optimality of **x** and **y** solutions*

If $\alpha = \beta = 1$ => standard complementary slackness conditions

Overview of the Schema

- Pick a primal *infeasible* solution **x**, and a dual *feasible* solution **y**, such that the slackness conditions are satisfied for chosen α and β (usually **x** = **0**, **y** = **0**).
- Iteratively improve the feasibility of **x** (integrally) and the optimality of **y**, such that the conditions remain satisfied, until **x** becomes feasible.

Proposition

An approximation guarantee of $\alpha\beta$ is achieved using this schema.

Proof:

$$\underbrace{\sum_{j=1}^{n} c_j x_j \le \alpha \sum_{j=1}^{n} \left(\sum_{i=1}^{m} a_{ij} y_i \right) x_j}_{\left({}^{c_i}/_{\alpha} \le \sum_{i=1}^{m} a_{ij}\, y_i\right)} = \underbrace{\alpha \sum_{i=1}^{m} \left(\sum_{j=1}^{n} a_{ij} x_j \right) y_i \le \alpha\beta \sum_{i=1}^{m} b_i y_i}_{\left(\sum_{j=1}^{n} a_{ij} x_j \;\le\; \beta b_i\right)}$$

$$\underbrace{\sum_{j=1}^{n} c_j x_j \le \alpha\beta \sum_{i=1}^{m} b_i y_i}_{(\textit{primal and dual condition})} \le \alpha\beta \cdot OPT_f \le \alpha\beta \cdot OPT$$

Set Cover(Recall)

Standard Primal Program

minimize $\sum_{\mathrm{s} \in \boldsymbol{S}} \boldsymbol{c}(\mathrm{S})\, \boldsymbol{x}_{\mathrm{s}}$

subject to $\sum_{\boldsymbol{e}:\mathrm{e}\, \in \mathrm{S}} \boldsymbol{x}_{\mathrm{s}} \geq 1, e \in U$

$\boldsymbol{x}_{\mathrm{s}} \in \{0,1\}, S \in \boldsymbol{S}$

The Dual Program

maximize $\sum_{\mathrm{e} \in \boldsymbol{U}} \boldsymbol{y}_{\mathrm{e}}$

subject to $\sum_{\mathrm{e}:\mathrm{e}\, \in \mathrm{S}} \boldsymbol{y}_{\mathrm{e}} \leq c(S), S \in \boldsymbol{S}$

$\boldsymbol{y}_{\mathrm{e}} \geq 0, e \in \boldsymbol{U}$

Relaxed Complementary Slackness Conditions

Definition: Set S is called tight if $\sum_{e:e\in S} y_e = c(S)$

Set $\alpha = 1, \beta = f$ (to obtain approximation factor f)
Primal Conditions – "Pick only tight sets in the cover"

$$\forall \mathrm{S} \in \boldsymbol{S}: \boldsymbol{x}_{\mathrm{S}} \neq \mathbf{0} \Rightarrow \sum_{\boldsymbol{e}:\boldsymbol{e}\in \mathrm{S}} \boldsymbol{y}_{\boldsymbol{e}} = \boldsymbol{c}(\mathrm{S})$$

Dual Conditions – "Each e, $\boldsymbol{y}_{\boldsymbol{e}} \neq \mathbf{0}$, can be covered at most $\boldsymbol{f}$ times"
- **trivially satisfied**

$$\forall \boldsymbol{e}: \boldsymbol{y}_{\boldsymbol{e}} \neq \mathbf{0} \Rightarrow \sum_{\boldsymbol{S}:\boldsymbol{e}\in \boldsymbol{S}} \boldsymbol{x}_{\boldsymbol{S}} \leq \boldsymbol{f}$$

Algorithm (Set Cover – factor f)

- Initialization: $\mathbf{x} = \mathbf{0}$, $\mathbf{y} = \mathbf{0}$.
- Until all elements are covered, do

  Pick an uncovered element e, and raise y_e until some set goes tight.

  Pick all tight sets in the cover and update **x**.

  Declare all the elements occuring in these sets as "covered".
- Output the set cover **x**.

Theorem

The algorithm can achieve an approximation factor of f.

Proof

- Clearly, there will be no uncovered and no overpacked sets in the end. Thus, primal and dual solutions will be feasible.
- Since they satisfy the relaxed complementary slackness conditions with $\alpha = 1, \beta = f$, the approximation factor is f.

Conclusion

- Combinatorial algorithms are greedy and local.
- LP-based algorithms are global.

Conclusion

- Combinatorial algorithms
  - Greedy algorithms Factor: Hn
  - Layering algorithms Factor: f
- LP-based algorithms
  - Rounding Factor: f, Hn
  - Primal–Dual Schema Factor: f