An Unconditional, Deterministic Polynomial Time Algorithm for Primarility Testing

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Based on

PRIMES is in P

By Manindra Agrawal, Neeraj Kayal and Nitin Saxena

Introduction

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Why are we interested in primes?

Primality Testing

- $\rightarrow$ Input: A positive number n in *binary*
- $\rightarrow$ Prime? Yes or No

Primality Testing

The algorithm we present is

- $\rightarrow$ Unconditional
- $\rightarrow$ Deterministic
- $\rightarrow$ Polynomial Time

Fermat's Little Theorem

For any prime number p, and any number a not divisible by p,

$$a^{p-1} = 1 \pmod{p}$$

- → Efficient to calculate $\ddot\smile$
- → However, many composites n also satisfy this for some a's $\ddot\frown$
- → *Carmichael Numbers*: $561, 1105, 1729, \ldots$

Other Approaches

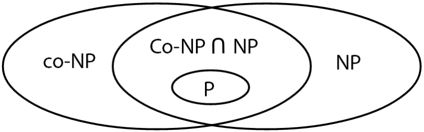
	Uncond	Det	Poly
Miller	×	✓	✓
Miller-Rabin	✓	×	✓
Solovay Strassen	✓	×	✓
APR*	✓	✓	×
Goldwasser & Kilian	×	×	✓

*APR = Adleman, Pomerance & Rumely

Computional Complexity

The problem is in

$$\mathtt{NP} \cap \mathtt{co\text{-}NP}$$



→ Why in $\mathtt{NP}$?

→ Why in $\mathtt{co\text{-}NP}$?

Primes in Times . . .

The New York Times (August 8, 2002) article

→ *Gödel Prize ('06)*

→ *Fulkerson Prize ('06)*

The Idea

The million dollar question

- Is n prime or composite?
- Is there a litmus test? YES!

Child's Binomial Theorem

- $a \in \mathbb{Z}$, $n \in \mathbb{N}$, $n \geq 2$ and $gcd(a, n) = 1$
- Then n is prime iff,

$$(x + a)^n = x^n + a \pmod{n}$$

The Litmus Test

Given n and a such that $gcd(a, n) = 1$ should $(x + a)^n = x^n + a \pmod{n}$?

- $\rightarrow$ If n is prime, then yes
- $\rightarrow$ If n is composite, then no

How do we prove it?

- $\rightarrow$ Substitute $(x + a)^n = x^n + \sum_{0<i<n} \binom{n}{i} x^i a^{n-i} + a^n$
- $\rightarrow$ $[(x + a)^n - x^n - a] \pmod{n} = 0$?

The Litmus Test: Proof

$$\rightarrow [x^n + \sum_{0<i<n} \binom{n}{i} x^i a^{n-i} + a^n - x^n - a] \pmod{n}$$

$$\equiv [\sum_{0<i<n} \binom{n}{i} x^i a^{n-i}] \pmod{n} + [a^n - a] \pmod{n}$$

$\rightarrow$ Since $a^n \pmod{n} = a$,

$$\equiv [\sum_{0<i<n} \binom{n}{i} x^i a^{n-i}] \pmod{n}$$

$\Rightarrow$: If n is prime

$$\rightarrow \ [\sum_{0<i<n} \binom{n}{i} x^i a^{n-i}] \pmod{n} = 0$$

$$\equiv \ \forall 0 < i < n, [\tfrac{n!}{i!(n-i)!}] \pmod{n} = 0$$

$\rightarrow$ $n - i < n$ and $i < n$ and n is prime

$\equiv$ No factor of n in denominator

$$\rightarrow \ [\tfrac{(i+1)\dot{(i+2)}\ldots n}{(n-i)!}] \pmod{n}$$

$$\equiv \ [(\tfrac{(i+1)\dot{(i+2)}\ldots(n-1)}{(n-i)!}) * n] \pmod{n}$$

$$\equiv \ 0$$

$\Leftarrow$: If n is composite

- $\rightarrow$ q: prime factor of n
- $\rightarrow$ $\exists k$ such that $q^k \parallel n$

$\Leftarrow$: If n is composite

- q: prime factor of n
- $\exists k$ such that $q^k \parallel n$

- Coefficient of $x^{n-q}a^q$ in $(x+a)^n$

$$\equiv \left[\left(\frac{n!}{(n-q)! \cdot q!}\right) x^{n-q} a^q\right] \pmod{n}$$

$$\equiv \left[\left(\frac{(n-q+1)\ldots(n)}{q!}\right) x^{n-q} a^q\right] \pmod{n}$$

$\Leftarrow$: If n is composite

$$[(\frac{(n-q+1)\dots(n)}{q!})] \pmod{n}$$

- $\rightarrow$ $[(\frac{(n-q+1)\dots(n)}{q!})] \pmod{n} \neq 0$
- $\rightarrow$ The only term q divides in the numerator is n
- $\rightarrow$ The only term q divides in the denominator is q
- $\rightarrow$ q^{k-1} is the highest power of q that divides $\binom{n}{q}$
- $\rightarrow$ $\therefore q^k \nmid \binom{n}{q} \Rightarrow n \nmid \binom{n}{q}$

$\Leftarrow$: If n is composite

$a^q \pmod{n}$

$\rightarrow \ gcd(a, n) = 1$

$\rightarrow \ gcd(a, q^k) = 1$

$\rightarrow \ gcd(a^q, q^k) = 1$

Outline

- → Given n, a and $gcd(a, n) = 1$
- → Calculate $f(x) := (x + a)^n - (x^n + a)$
- → As $f(x) \pmod{n} = 0$
- → $\sum_{0<i<n} \binom{n}{i} X^i a^{n-i}$ - each term should be zero
- → Computation of n coefficients
- → $\Omega(n)$: horribly inefficient!

AKS: The Idea

→ Can we reduce the number of coefficients to be calculated?

$$x^n + \sum_{0<i<n} \binom{n}{i} x^i a^{n-i} + a^n \pmod{n}$$

$$\Downarrow$$

$$x^r + \sum_{0<i<r} \binom{r}{i} x^i a^{n-i} + a^n \pmod{n}$$

AKS: The Idea

- → The algorithm finishes in polynomial time
- → Only r number of calculations
- → For a small r, check if

$$(x+a)^n = x^n + a \pmod{x^r - 1, n}$$

(we refer to this as the AKS Equation)

- → Necessary and Sufficient!

Preliminaries

Group

Integers $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$ under addition forms a *group*, denoted $(\mathbb{Z}, +)$.

- $\rightarrow$ Closure: $a + b \in \mathbb{Z}$
- $\rightarrow$ Associativity: $(a + b) + c = a + (b + c)$
- $\rightarrow$ Identity element: $z + 0 = z$
- $\rightarrow$ Inverse element: $n + (-n) = 0$

$(\mathbb{Z}, +)$ is also an *abelian group* since it satisfies:

- $\rightarrow$ Commutativity: $a + b = b + a$

More Group Examples

For any $n \in \mathbb{N}^+$

- $\rightarrow$ Integers modulo n forms a group under addition modulo n
- $\rightarrow$ Identity element is 0
- $\rightarrow$ Inverse element of x is $(n - x)$ mod n

For $n = 6$, the *abelian* group is $\{\bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{4}, \bar{5}\}$

$$\bar{1} + \bar{2} = \bar{3}$$
$$\bar{3} + \bar{4} = \bar{1}$$
$$\bar{5} + \bar{1} = \bar{0}$$
$$\cdots$$

More Group Examples

For any prime p,

- Integers modulo p is a multiplicative group
- Elements: integers 1 to $p - 1$
- Group operation: multiplication modulo p
- It's an abelian group, too

For example, if $p = 5$, group elements are $1, 2, 3, 4$

More Group Examples

When $p = 5$, the table of inverse elements:

$\times$	1	2	3	4
1	1	2	3	4
2	2	4	1	3
3	3	1	4	2
4	4	3	2	1

It is a cyclic group since the whole group can be generated by 2:

$$2^1 = 2, 2^2 = 4, 2^3 = 3, 2^4 = 1$$

Ring

Integers modulo n form a *ring* under modular *add* and *mult*, denoted $\mathbb{Z}_n = \{\bar{0}, \bar{1}, \ldots, \overline{n-1}\}$

- $\rightarrow$ Abelian Additive Group: $\mathbb{Z}_n$ is an abelian group under modular addition
- $\rightarrow$ Mult. Closure: $x \cdot y \in \mathbb{Z}_n$
- $\rightarrow$ Mult. Associativity: $(x \cdot y) \cdot z = x \cdot (y \cdot z)$
- $\rightarrow$ Mult. Identity: $x \cdot \bar{1} = x$
- $\rightarrow$ Distributivity: $x \cdot (y + z) = (x \cdot y) + (x \cdot z)$

Ring Example: Polynomial Ring

The polynomial ring, $K[x]$, in x over a ring K is the set of polynomials in x, of the form

$$c_m x^m + c_{m-1} x^{m-1} + \cdots + c_2 x^2 + c_1 x + c_0$$

where $c_i \in K$ and $x, x^2, \ldots$ are formal symbols

- $+$: Polynomial addition
- $\times$: Polynomial multiplication

Concretely, all polynomials over ring $\mathbb{Z}_n$ (denoted $\mathbb{Z}_n[x]$) form a polynomial ring

Field

- Intuitively, a field F is a generalization of concept of $\mathbb{R}$:

$$\text{We can do } +, -, \times, \div \text{ in } F$$

- Formally, a field is a ring whose nonzero elements form an abelian group under $\times$
- $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ are all fields

Field Example: Prime Field

For any prime p, integers modulo p form a field called *prime field*, denoted F_p

Addition in F_5

$+$	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

Multiplication in F_5

$\times$	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	4
2	0	2	4	1	3
3	0	3	1	4	2
4	0	4	3	2	1

Irreducible Polynomial

- $x^2 - 1$ is reducible over $\mathbb{Z}$ since $x^2 - 1 = (x-1)(x+1)$
- $x^2 - 5$ is irreducible over $\mathbb{Q}$ but reducible over $\mathbb{R}$ since $x^2 - 5 = (x - \sqrt{5})(x + \sqrt{5})$
- $x^2 + 1$ is irreducible over $\mathbb{Q}$ but reducible over F_2
- In $F_2[x]$, $(x+1)^2 = x^2 + 2x + 1 = x^2 + 1$

Irreducible Polynomial

- $\rightarrow$ If p is prime and $h(x)$ is a polynomial of degree d and irreducible over F_p, then $F_p[x]/(h(x))$ is a finite field of order p^d
- $\rightarrow$ Two fields of order 8 are $F_2[x]/(x^3+x+1)$ and $F_2[x]/(x^3+x^2+1)$

Modular Operations on Polynomials

→ We can calculate $P(x)$ mod $Q(x)$ using polynomial long division:

$$
\begin{array}{rrrrrr}
 & & x^5 + x^3 & & + x & \\
\cline{2-5}
x^2 - 1 \big) & x^7 & & & + 6x & - 7 \\
 & -x^7 + x^5 & & & & \\
\cline{2-3}
 & x^5 & & & & \\
 & -x^5 + x^3 & & & & \\
\cline{2-4}
 & & x^3 & & + 6x & \\
 & & -x^3 & & + x & \\
\cline{3-5}
 & & & & 7x &
\end{array}
$$

Modular Operations on Polynomials

→ So $x^7 + 6x - 7 = 7x - 7 \pmod{x^2 - 1}$

→ $f(x) = g(x) \pmod{h(x), n}$ means
$f(x) = g(x)$ in $\mathbb{Z}_n[x]/(h(x))$

Cyclotomic Polynomial

- → A n^{th} cyclotomic polynomial $\Phi_n(x)$ is the unique irreducible polynomial with integer coefficients
- → Divisor of $x^n - 1$, not a divisor of $x^k - 1$ for any $k < n$

$$\Phi_n(x) = \prod_{\substack{1 \leq k \leq n \\ \gcd(k,n)=1}} \left(x - e^{2i\pi\frac{k}{n}}\right)$$

Cyclotomic Polynomial: Examples

- $\Phi_{15}(x) = x^8 - x^7 + x^5 - x^4 + x^3 - x + 1$
- $\Phi_{11}(x) =$
  $x^{10} + x^9 + x^8 + x^7 + x^6 + x^5 + x^4 + x^3 + x^2 + x + 1$

Order of a modulo r

- Given $\gcd(a, r) = 1$, the order of a modulo r is the smallest number k such that $a^k = 1 \pmod{r}$
- It is denoted as $o_r(a)$

Order of a modulo r

- $\rightarrow$ Why does k exist?
- $\rightarrow$ For a given r, $\{a \mid (a, r) = 1 \land a < r\}$ forms a finite abelian group under multiplication modulo r
- $\rightarrow$ For a specific a, $\exists k_1 < k_2$, such that $a^{k_2} = a^{k_1} \pmod{r}$.
  So, $a^{k_2 - k_1} = 1 \pmod{r}$

Order of a modulo r: Example

$\rightarrow$ For $r = 20$, $a = 7$, $o_{20}(7) = 4$ since

$$
\begin{aligned}
7^2 &= 49 = 9 \pmod{20} \\
7^3 &= 343 = 3 \pmod{20} \\
7^4 &= 2401 = 1 \pmod{20}
\end{aligned}
$$

The Algorithm

The Main Algorithm

Input: integer $n > 1$

(1) Preliminary test

(2) Find a suitable r

(3) Search for non co-prime elements

(4) `if` $n \leq r$, `return` PRIME

(5) `for` $a = 1, 2, \ldots, \lceil \sqrt{r} \log n \rceil$ `do`

(6) `if` $(X - a)^n \neq X^n - a \pmod{X^r - 1, p}$ `then`

`return` COMPOSITE

(7) `return` PRIME

The Process

(1) Preliminary test

If n is perfect power

$\rightarrow$ Given n, if $n = a^b (b > 1)$, n is composite

$\rightarrow$ $b < \log n + 1$

Then for every b, we can find such a using binary search

The Process

(2) Find suitable r

Find the smallest r such that $o_r(n) > (\log n)^2$

$\rightarrow$ Recall, order $o_r(n)$ is smallest j such that $n^j = 1 \pmod r$

`for` $q = 1, 2, \cdots, \lceil (\log n)^5 \rceil$ `do`
 `if` $n^j \neq 1 \pmod q$ `for` $j = 1, 2, \ldots, \lceil (\log n)^2 \rceil$
  $r = q$

(Why $r \leq \lceil (\log n)^5 \rceil$? We shall see later!)

The Process

(3) Search for non co-prime elements

If $\gcd(a, n) > 1$ for some $a \leq r$, COMPOSITE

Use Euclidean algorithm for each a to check if $\gcd(a, n) > 1$
If such an a exists, then n is composite

The Process

(5--6) Main loop

$$
\begin{array}{l}
\texttt{for } a = 1 \texttt{ to } \lceil \sqrt{r} \log n \rceil \texttt{ do} \\
\quad \text{if } (X+a)^n \neq X^n + a \pmod{X^r - 1, n} \texttt{ then} \\
\quad\quad \texttt{return } \text{COMPOSITE}
\end{array}
$$

Use standard mod calculation with fast exponentiation

Putting it all together

Input: integer $n > 1$

(1) `if` $n = a^b$, for $a, b \geq 2$ && $b < \log n + 1$ `then`
 `return` COMPOSITE

(2) choose smallest r such that $o_r(n) > (\log n)^2$

(3) `if` $\exists \, \gcd(a, n) < n$ for some $a < r$
 `return` COMPOSITE

(4) `if` $n \leq r$, `return` PRIME

(5) `for` $a = 1, 2, \ldots, \lceil \sqrt{r} \log n \rceil$ `do`

(6) `if` $(X + a)^n \neq X^n + a \pmod{X^r - 1, p}$ `then`
 `return` COMPOSITE

(7) `return` PRIME

Time Complexity Analysis

Shruti Tople

Arithmetic Computation & $\widetilde{O}$

- $\rightarrow$ If a and b are two positive integers, each with no more than m digits in binary
- $\rightarrow$ $+$ and $-$ take $O(m)$ bit operations
- $\rightarrow$ $\times$ takes $O(m(\log m)^{O(1)})$

We define $\widetilde{O}(m) = O(m(\log m)^{O(1)})$

- $\rightarrow$ For two d degree polynomials with m bit coefficients, multiplication takes $\widetilde{O}(d \cdot m)$

Complexity Analysis

(1) Given n, if $n = a^b (b > 1)$, n is composite

$\rightarrow$ Bound on b: $b < \log n + 1 \Rightarrow O(\log n)$

$\rightarrow$ For every b, find a using binary search $\Rightarrow O(\log n)$

$\rightarrow$ To compute $a^b \Rightarrow \widetilde{O}(\log n)$

Complexity of Step 1: $\widetilde{O}((\log n)^3)$ bit operations

Complexity Analysis

(2) Find the smallest r such that $o_r(n) > (\log n)^2$

$\texttt{for}\ q = 1, 2, \cdots, \lceil (\log n)^5 \rceil\ \texttt{do}$

$\quad \texttt{if}\ n^j \neq 1 \pmod{q}\ \texttt{for}\ j = 1, 2, \cdots, \lceil (\log n)^2 \rceil$

$\quad\quad r = q$

→ First `for` loop $\Rightarrow O(r)$; worst case $O((\log n)^5)$

→ Second `for` loop $\Rightarrow \widetilde{O}((\log n)^2)$

Complexity of Step 2: $\widetilde{O}(r(\log n)^2) = \widetilde{O}((\log n)^7)$

Complexity Analysis

(3) If $\gcd(a, n) > 1$ for some $a \leq r$, n is COMPOSITE

$\rightarrow$ Euclidean algorithm complexity $\Rightarrow O(\log n)$

$\rightarrow$ As $a \leq r$, in worst case need $O(r)$ computation

This can be done in $O(r(\log n)) = O((\log n)^6)$

Complexity Analysis

(5) $\texttt{for}\ a = 1$ to $\lceil \sqrt{r} \log n \rceil$ $\texttt{do}$

(6) $\quad \texttt{if}\ (X+a)^n \neq X^n + a \pmod{X^r - 1, n}$ $\texttt{then}$

$\quad\quad \texttt{return}$ COMPOSITE

We have, a degree r polynomial with $\log n$ bits

$\rightarrow$ Bitwise multiplication $\Rightarrow \widetilde{O}(r(\log n)^2)$

$\rightarrow$ `for` loop runs from 1 to $\sqrt{r} \log n$

$\rightarrow$ Now, the complexity is: $\widetilde{O}(r(\log n)^2 \cdot \sqrt{r} \log n)$

$$= \widetilde{O}(r^{\frac{3}{2}}(\log n)^3) = \widetilde{O}((\log n)^{\frac{21}{2}})$$

Complexity Analysis

Overall complexity

- $\rightarrow$ Step 1: $\widetilde{O}((\log n)^3)$
- $\rightarrow$ Step 2: $\widetilde{O}(r(\log n)^2)$
- $\rightarrow$ Step 3: $O(r(\log n))$
- $\rightarrow$ Final loop: $\widetilde{O}((\log n)^{\frac{21}{2}})$

Complexity of the final loop dominates all others

Hence, overall complexity: $\widetilde{O}((\log n)^{\frac{21}{2}})$

Proof of Correctness

AKS Theorem

For the smallest r such that $o_r(n) > (\log n)^2$

n is prime iff

- $\rightarrow$ n is not a perfect power,
- $\rightarrow$ n does not have any prime factor $\leq r$,
- $\rightarrow$ $(x+a)^n = x^n + a \mod (n, x^r - 1)$ for each integer a, $1 \leq a \leq A = \lceil \sqrt{r} \log n \rceil$

If n is prime

- → If n is prime, steps (1) and (3) can never return COMPOSITE
- → The `for` loop can not return COMPOSITE either
- → Hence the algorithm will output PRIME

We are only left with the other side of the proof!

If the Algorithm Returns PRIME

Proof by contradiction

- → Let's assume n is composite
- → Thus, there exists a prime p such that $p|n$

We assume

- → n is not a perfect power
- → n does not have any prime factor $\leq r$

If the Algorithm Returns PRIME

The master plan:

- $\rightarrow$ We show that there exists a *suitable* r

- $\rightarrow$ We construct a nice group $\mathbb{G}$ assuming $p|n$
- $\rightarrow$ We prove a contradiction on the size of $\mathbb{G}$
  - $\Rightarrow$ There is no such $\mathbb{G}$
- $\rightarrow$ Hence, n is not composite

We assume $\operatorname{lcm}\{1, \cdots, m\} \geq 2^m$ for $m \geq 7$

Existence of a Suitable r

There exists an $r \leq \max\left(3, \lceil (\log n)^5 \rceil\right)$ such that

$$o_r(n) > (\log n)^2$$

- $\rightarrow$ When $n = 2$, $r = 3$. We assume $n > 2$, thus $\lceil (\log n)^5 \rceil > 10$
- $\rightarrow$ Consider $\{r_1, r_2, \cdots, r_t\}$ such that either $o_r(n) \leq (\log n)^2$ or $r_i | n$
- $\rightarrow$ Thus, every r_i divides

$$n \cdot \prod_{i=1}^{\lceil (\log n)^2 \rceil} (n^i - 1) < n^{(\log n)^4} \leq 2^{(\log n)^5}$$

Existence of a Suitable r

But the lcm of the first $\lceil (\log n)^5 \rceil$ numbers is at least $2^{\lceil (\log n)^5 \rceil}$

Thus, $\exists s \leq \lceil (\log n)^5 \rceil$, such that $s \notin \{r_1, \cdots, r_t\}$

- $\rightarrow$ If $\gcd(s, n) = 1$, then $o_s(n) > (\log n)^2$
- $\rightarrow$ If $\gcd(s, n) > 1$, then since $s \nmid n$ and $(s, n) \in \{r_1, \cdots, r_t\}$, $r = \frac{s}{\gcd(s,n)} \notin \{r_1, \cdots, r_t\}$ and so $o_r(n) > (\log n)^2$

Find a Nice Group $\mathbb{G}$

For each integer a, $1 \leq a \leq A$,

→ We know

$$(x+a)^n = x^n + a \pmod{x^r - 1, n}$$

→ $p|n$, hence

$$(x+a)^n = x^n + a \pmod{x^r - 1, p}$$

→ Let $h(x)$ be an irreducible factor of $\Phi_r(x)$ $\pmod{p}$ (i.e. in $(\mathbb{Z}/p\mathbb{Z})[x]$), then

$$(x+a)^n = x^n + a \pmod{h(x), p}$$

Find a Nice Group $\mathbb{G}$

- Given $\mathbb{F} = \mathbb{Z}[x]/(p, h(x))$, non-zero elements of $\mathbb{F}$ form a cyclic group of order $p^m - 1$
- Let H be the multiplicative group modulo $(x^r - 1, p)$ generated by $x, x+1, x+2, \cdots, x+A$
- Let $\mathbb{G}$ be the (multiplicative) subgroup of $\mathbb{F}$ generated by $x, x+1, x+2, \cdots, x+A$
- All the elements of $\mathbb{G}$ are non-zero

Bounds on $|\mathbb{G}|$

$g(x) = \prod_{0 \le a \le A} (x+a)^{e_a} \in H$, then

$$
\begin{aligned}
g(x)^n &= \prod_a ((x+a)^n)^{e_a} \pmod{x^r - 1, p} \\
&= \prod_a (x^n + a)^{e_a} \pmod{x^r - 1, p} \\
&= g(x^n) \pmod{x^r - 1, p}
\end{aligned}
$$

Bounds on $|\mathbb{G}|$

Define S to be the set of positive integers k for which $g(x^k) = g(x)^k \pmod{x^r - 1, p}$ for all $g \in H$

- $p, n \in S$

A few properties of S:

- If $a, b \in S$, $ab \in S$ (Lemma 1)
- If $a, b \in S$ and $a = b \pmod{r}$, then $a = b \pmod{|\mathbb{G}|}$ (Lemma 2)

Upper Bound on $|\mathbb{G}|$

- Let R be the subgroup of $(\mathbb{Z}/r\mathbb{Z})^*$ generated by n and p
- There exist more than $|R|$ integers of the form $n^i p^j$ with distinct $0 \leq i, j \leq \sqrt{|R|}$
- Two of them must be congruent $\pmod{r}$
- Say, $n^i p^j = n^I p^J \pmod{r}$
- $|\mathbb{G}| \leq |n^i p^j - n^I p^J| \leq (np)^{\sqrt{|R|}-1} \leq n^{2\sqrt{|R|}-1}$
- If $n/p \in S$, $|\mathbb{G}| \leq n^{\sqrt{|R|}-1}$

Lower Bound on $|\mathbb{G}|$

- $\rightarrow$ The products $\prod_{a \in T}(x+a)$ give distinct elements of $\mathbb{G}$ for every proper subset T of $\{0, 1, 2, \cdots, \lceil \sqrt{|R|} \log n \rceil\}$
- $\rightarrow$ $|\mathbb{G}| \geq 2^{\lceil \sqrt{|R|} \log n \rceil + 1} - 1 > n^{\sqrt{|R|}} - 1$

The upper and lower bounds conflict, thus making our only assumption wrong

There exists no such $\mathbb{G}$

Hence, n is not composite, completing the proof of correctness

Supplementary Material

Supplementary Material

If $a, b \in S$, $ab \in S$

$\rightarrow$ If $g(x) \in H$, $g(x^b) = g(x)^b \pmod{x^r - 1, p}$

$\rightarrow$ Replacing x by x^a, we get

$g((x^a)^b) = g(x^a)^b \pmod{(x^a)^r - 1, p}$ and

hence $\pmod{x^r - 1, p}$

$\rightarrow$

$$
\begin{aligned}
g(x)^{ab} &= g((x)^a)^b && \ldots (a \in S) \\
&= g((x^a)^b) && \ldots (b \in S) \\
&= g(x^{ab}) \pmod{x^r - 1, p}
\end{aligned}
$$

Supplementary Material

If $a, b \in S$ and $a = b \pmod{r}$, then $a = b \pmod{|\mathbb{G}|}$

$\rightarrow$ $(x^r - 1) | (x^{a-b} - 1)$ and $(x^{a-b} - 1) | (x^a - x^b)$

$\rightarrow$ $(x^a - x^b) | (g(x^a) - g(x^b))$

$\rightarrow$ $(x^r - 1) | (g(x^a) - g(x^b))$

$\rightarrow$ $g(x) \in H$, then $g(x)^a = g(x)^b \pmod{x^r - 1, p}$

$\rightarrow$ If $g(x) \in \mathbb{G}$, $g(x)^{a-b} = 1$ in $\mathbb{F}$

$\rightarrow$ Since $\mathbb{G}$ is cyclic, taking generator g, $|\mathbb{G}|$ divides $a - b$

Statements Not Proved

- $\rightarrow$ $\operatorname{lcm}\{1, \cdots, m\} \geq 2^m$ for $m \geq 7$
- $\rightarrow$ $n/p \in S$
- $\rightarrow$ Two distinct polynomials of the form $\prod_a (x + a)$ of degree $< |R|$ will map to different elements of $\mathbb{G}$

Limitations and Future Work

Usefulness

$$996546876313652827462 8451$$

AKS in SAGE* takes ≈ 70 min for the above number!

*(Software for Algebra and Geometry Experimentation)

Comparison

Practical alternatives

- → APR primality runs in $\widetilde{O}((\log n)^{(\log\log\log n)})$ and yet performs better than AKS
- → Miller-Rabin and other randomized algorithms, which takes average time $\widetilde{O}(\log n)^3$, are used in practice

Agrawal's Conjecture

- → The `for` loop in the algorithm (in step 5) runs $\lceil(\sqrt{r}\log n)\rceil$ times
- → This can be reduced assuming the following conjecture:

If r is a prime number that does not divide n and if $(x+1)^n = x^n + 1 \pmod{x^r - 1, n}$ then either n is prime or $n^2 = 1 \pmod{r}$

Agrawal's Conjecture: Consequences

- $\rightarrow$ We can modify the algorithm to search for an r which does not divide $n^2 - 1$
- $\rightarrow$ Such an r exists in $[2, 4 \log n]$ (product of prime numbers less than x is at least e^x)
- $\rightarrow$ Verifying the congruence takes $\widetilde{O}(r(\log n)^2)$.
- $\rightarrow$ Overall complexity: $\widetilde{O}(\log n)^3$

Agrawal's Conjecture: Progress

- 2003: Lenstra and Pomerance gave a heuristic argument that suggested that the conjecture is false.
- 2005: A group at UT Austin proved that the conjecture is true if $r > n/2$

Other Improvements

Possible improvements in implementation

→ Mapping the polynomial rings onto integer rings

→ Using suitable libraries (NTL better than LiDIA)

Summary

- Efficient to work with Child's Binomial Theorem by reducing its degree by a factor r
- Use this for primality test which runs in polynomial time
- Possible improvements

Take Away

The ground breaking AKS Primality Test is

- $\rightarrow$ unconditional
- $\rightarrow$ deterministic
- $\rightarrow$ polynomial time

References

1. Granville, Andrew. "It is easy to determine whether a given integer is prime." Bulletin of the American Mathematical Society 42.1 (2005): 3-38.
2. Student Talks by S Ramprasad The AKS Primality Test
3. Lang, Serge. Undergraduate algebra. Springer, 2005.

Thank You!
Questions?