Summary

Following is an outline of the algorithm discussed in “A randomized polynomial time algorithm for approximating the volume of a convex body”. Input to the algorithm is an oracle (which decides the membership of a point in a fixed convex body), two spheres (of radius depending only on dimension of space) between which the body is promised to be positioned and an error parameter ‘e’. Algorithm outputs in polynomial time (polynomial in dimension of the space), the correct volume of the body upto (1+- e) factor, with probability at least ¾.
It may be noted that there can be no polynomial time deterministic algorithm given all the above inputs.

Algorithm

Let K be the convex body in R^n whose volume is to be found. In the algorithm, all the convex bodies are given by a “well-guaranteed membership oracles”, that is, we will be given a sphere contained in the body, a sphere containing the body and a black box, which presented with any point x in space, either replies that x is in the convex body or that it is not. On applying affine transformation, the convex body is "well-rounded". That is, the body contains the unit ball with the origin as center and is contained in a concentric ball of radius $r = \sqrt{n}(n+1)$ where n is the dimension of the body. So we have $B \subseteq K \subseteq rB$.
Let $\rho = 1 - 1/n$. Let $k = \left|log_{1/\rho} r\right|$ and for i=0, 1, 2, …, k, let $\rho_i = max\{\rho^i r, 1\}$. The algorithm will find for i=1, 2, …, k an approximation to the ratio

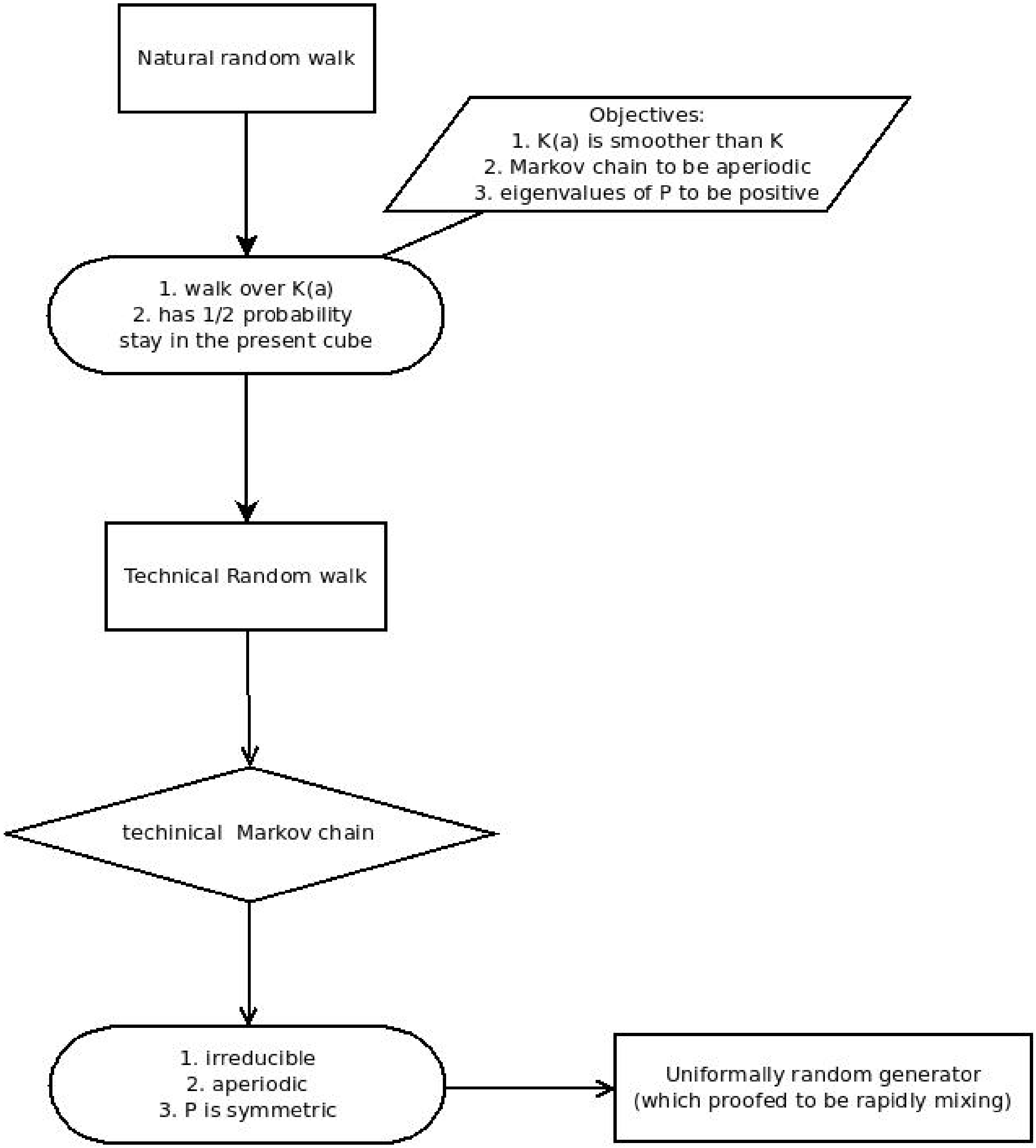
$$\frac{Vol_n(\rho_i K \cap rB)}{Vol_n(\rho_{i-1} K \cap rB)}$$

The ratio will be found by a sequence of “trials”. In each trail, we first do the technical random walk on $K_{i-1} = \rho_{i-1} \cap rB$ for τ steps. Suppose we are in cube $C = \{x : q_i\delta \le x \le (q_i + 1)\delta\}$ after τ steps. We pick randomly a point x_0 in cube C. If $x_0 \in K_{i-1}$, then we declare the trial a proper trial and check to see if $x_0 \in K_i$. If it does, we declare the trial a success. This completes the trial. We repeat until we have made m proper trials and we keep track of the ratio of the number of successes to m.
Clearly this together with the fact that $K_k = K$ and the volume of $K_0 = rB$ is known in closed form gives us the volume of K.

Random walk

In the algorithm, what we need is a uniformly random generator, the diagram below shows the technical random walk we are using, and why it is a uniformly random generator.

Later on we will show that the Markov chain for this technical random walk actually is rapidly mixing, which means that it can reach a steady distribution in polynomial time steps.



Proof of correctness

As described earlier, the algorithm approximates the volume of the convex body by approximating a series of volume ratios Vol_n(K_i)/Vol_n(K_(i-1)), and these ratios are actually related to the ratio of Pr(success trial)/Pr(proper trial). Therefore, the proof of correctness starts by relating the volume ratio to the probability ratio. The authors then show that the probability ratio approximates the volume ratio within a certain bound with a probability of at least ¾.

$$Pr(proper\ trial) = \sum_{C \in W} Pr(proper\ trial | walk\ ends\ in\ C) * Pr(walk\ ends\ in\ C)$$
$$\leq \sum_{C \in W} \left(a_C + \frac{N_C^B}{N_C}\right)\left(\frac{1}{|W|} + \left(1 - \frac{1}{10^{17} n^{19}}\right)^{\tau}\right)$$

Based on basic probability theory, the probability of proper trial is equal to the first expression where W is the set of all cubes that weakly intersects the convex body K_(i-1). The first expression is upper bounded by the second equation where N^B_C/N_C and (1-1/(10^17*n^19))^t are the error values. Simplifying the expression and applying the same argument to the convex body K_i gives us the following two expressions respectively.

$$\frac{Vol_n(K_{i-1})}{|\mathrm{W}|\delta^n}\left(1 - \frac{\epsilon}{100k}\right) \leq \Pr(proper\ trial) \leq \frac{Vol_n(K_{i-1})}{|\mathrm{W}|\delta^n}\left(1 + \frac{\epsilon}{100k}\right)$$

$$\frac{Vol_n(K_i)}{|\mathrm{W}|\delta^n}\left(1 - \frac{\epsilon}{100k}\right) \leq \Pr(success \cap proper\ trial) \leq \frac{Vol_n(K_i)}{|\mathrm{W}|\delta^n}\left(1 + \frac{\epsilon}{100k}\right)$$

The probability ratio, p can then be calculated by dividing the two expressions above to give the following expression.

$$v\left(1 - \frac{\epsilon}{100k}\right)\left(1 + \frac{\epsilon}{100k}\right)^{-1} \leq p \leq v\left(1 + \frac{\epsilon}{100k}\right)\left(1 - \frac{\epsilon}{100k}\right)^{-1}, v = Vol_n(K_i)/Vol_n(K_{i-1})$$

The expression above gives the error of a single volume estimate. Multiplying the expression by itself k times gives the error of the convex body volume estimate, and the error can be shown to be bounded by the following expression with Hoeffding's inequality where V is the approximated volume. The prove is now complete.

$$(1 - \epsilon) \leq \frac{V}{Vol_n(K)} \leq (1 + \epsilon) \text{ with a probability of } at\ least \frac{3}{4}$$

Rapidly mixing Markov chain

Referring back to some properties of Markov Chain, it is known that any finite, ergodic Markov Chain converges to a unique stationary distribution π after an infinite number of steps, that is:

$$\lim_{s \to \infty} p_{i,j}^{(s)} = \pi_j \forall i, j$$
$$\sum_j \pi_j = 1$$

By formal definition, the time taken by a particular Markov Chain to converge to its stationary distribution is called mixing time. It is measured in terms of the *total variation distance* between the distribution at time s and the stationary distribution:

$$\left\| p^s, \pi \right\|_{tv} = \max_{i \in \Omega} \frac{1}{2} \sum_{j \in \Omega} \left| p_{i,j}^s - \pi_j \right|$$

(Ω is the set of all states)

This total variation distance is introduced to reflect the fact that it is not possible to obtain the stationary distribution by running infinite number of steps and a small value $\varepsilon > 0$ is added to relax the convergent condition for the mixing time $\tau(\varepsilon)$ as below:

$$\tau(\varepsilon) = \min\{s : \left\| p^{s'}, \pi \right\|_{tv} \le \varepsilon, \forall s' \ge s\}$$

From the definition of mixing time, a Markov Chain is called rapidly mixing if the mixing time $\tau(\varepsilon)$ is O(poly(log(N/ε))), N is the number of states. It means that if N is exponential in problem size n, $\tau(\varepsilon)$ is actually bounded by O(poly(n)). The result obtained in the paper about the minimum required mixing time τ complies with this definition and the total variation distance is bounded with such a small value that guarantees the distribution received from the proposed Markov Chain is close enough to uniform distribution:

$$s = \tau = \left\lceil 10^{17} n^{19} \log\left(\left(\frac{3r}{\delta} \right)^n \frac{300k}{\varepsilon} \right) \right\rceil$$

$$\left\| p^s, \pi \right\|_{tv} \le \frac{1}{2} \frac{\varepsilon}{300k}$$

(ε here is the predefined accuracy of estimating the volume of convex body)

Proof for rapidly mixing

Using the result of Sinclair and Jerrum in present context, authors show that second largest eigenvalue of the transition matrix is bounded from above by **1-y(S)/4n**, where y(S) for a set is defined below.

Assume a set S with size less than N/2.
Let Sb be the set of boundary cubes of S, (S,barS) be the set of faces between cubes in S and cubes in barS, Si be the interior cubes of S. Then **y(S) = |(S,barS)|/|S|**.
Authors prove the following relation, **using the properties of the convex body K(alpha)**:
|Sb| <= 2n |(S,barS)| + 18|Si|.

This leads to following case, applicable to smooth version of K(alpha), called KK.
Case 1: Many boundary cubes: |Sb|/|S| >= 18.5/19. Then y(S) > 1/4n. Intuitively, more boundary cubes and less inside cubes means more ‘exposed’ intersection (S,barS) between boundary cubes and inside cubes of barS.
Case 2: |Sb|/|S| < 18.5/19. This means |Si|/|S| >= 1/38.
This means there are considerably many inside cubes. But this in turn can mean that intersection of Si with bar(S) is a lot and we must exploit it. Key idea: **isoperimetric inequality**

for euclidean space, which says that if volume of a region is large, then its surface area is large.
Let region R be defined to be intersection of KK with S. Let T be its boundary. Isoperimetric inequality ensures that |T| is comparable to |S|. Now, T is made up of: T1 (part that does not intersect KK) and T2(part that intersects KK). Let T3 be region of KK that is not T2, and bar(T) be the boundary of complement of R inside KK.
Clearly, T1 is a part of boundary between (S,barS). Using this, we have following cases.
Case 2.1: If Vol(T1) > Vol(T)/2 or Vol(T3)<Vol(bar(T))/2, then T1 is comparable to T and one gets $y(S) > 1/4000n^3$.
Case 2.2: If Vol(T2)>Vol(T)/2 and Vol(T3)> Vol(bar(T))/2. Now recall that T2 and T3 make up whole of boundary of KK. Next idea is to observe a yet another **isoperimetric inequality on curved space**: boundary of KK. This isoperimetric inequality says that if a region on boundary of KK is big, but not too big, then the boundary of this region has large perimeter. Using this idea, one realizes that boundary of T2 (T1; they have same boundary) is comparable to |S|. Hence one gets $y(S) > d^2/2400n^{7/2}$, since this boundary itself contributes to the (S,barS).

Conclusion

The algorithm provides a randomized polynomial time algorithm for a problem that has provably no polynomial time deterministic algorithm. This gives an interesting picture of the ‘oracle based’ computation. As application, assuming the existence of oracle, the algorithm gives us a way to perform integration of convex functions convex bodies, or looking at volume of solution space in linear programming. If oracle is easy to construct, then probably a deterministic algorithm is possible as well, since the body must be simple enough for this to hold. On the other hand, the proof of rapid mixing markov chain shows the power of the result of Sinclair and Jerrum, which widens the scope for developing techniques to prove rapid mixing.

The algorithm has been improved considerably, dropping the computational time from $O(n^{23})$ to $O(n^4)$.